Whitney Duals of Geometric Lattices

Rafael S. González D'León
Department of Mathematics
University of Kentucky

Josh Hallam
Department of Mathematics
Wake Forest University

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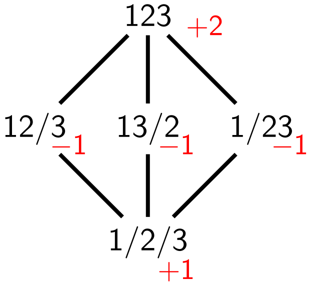
The Whitney Numbers

Throughout the talk we will assume that P is a graded poset with a minimum element $\hat{0}$ (but not necessarily a $\hat{1}$) and rank function ρ. The **(one–variable) Möbius function** $\mu : P \to \mathbb{Z}$ is defined by

$$\sum_{x \leq y} \mu(x) = \delta_{\hat{0},y}.$$

Let Π_n be the lattice of set partitions of $[n]$ with cover relation $\pi \lessdot \sigma$ if σ can be obtained from π by merging two blocks together.

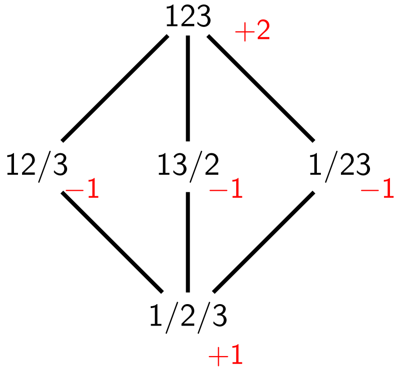


Π_3

The Whitney Numbers

The k^{th}-**Whitney Number of the first kind** of P is defined by

$$w_k(P) = \sum_{\rho(x)=k} \mu(x).$$



Π_3

$w_2(\Pi_3) = 2$

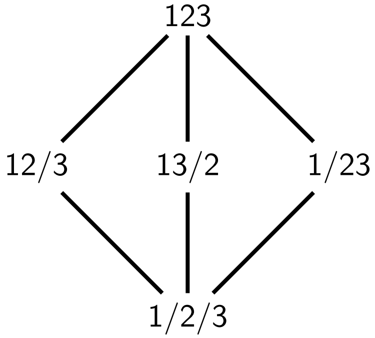
$w_1(\Pi_3) = -3$

$w_0(\Pi_3) = 1$

The Whitney Numbers

The k^{th}-**Whitney Number of the second kind** of P is defined by

$$W_k(P) = |\{x \in P \mid \rho(x) = k\}|.$$



Π_3

$W_2(\Pi_3) = 1$

$W_1(\Pi_3) = 3$

$W_0(\Pi_3) = 1$

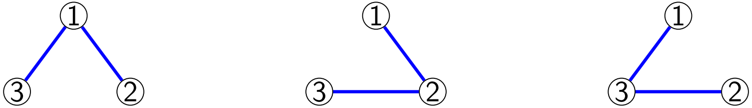
The Whitney Numbers

The Whitney numbers appear in several aspects of combinatorics. For example,

- The Stirling number of the first and second kind are the Whitney numbers of the first and second kind of the partition lattice Π_n
- The coefficients of the chromatic polynomial of a graph are Whitney numbers of the first kind.
- For a real affine hyperplane arrangement, the number of bounded regions is the sum of the Whitney numbers of the first kind of its intersection poset. The total number of regions is the sum of the absolute values of these numbers.

A Poset of Increasing Spanning Forests

Let T be a tree with vertices labeled by distinct integers. We say T is an **increasing tree** if the labels along any path starting at the smallest vertex increase.

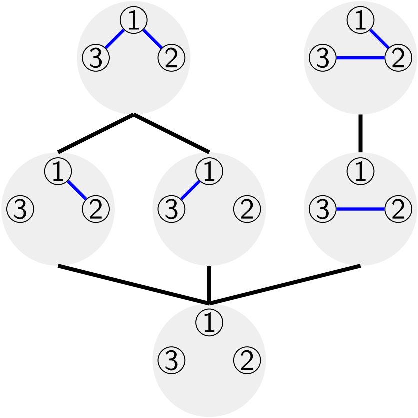


Increasing | Increasing | Not Increasing

A **increasing spanning forest** is a forest with vertex set $[n]$ such that each connected component is an increasing tree.

A Poset of Increasing Spanning Forests

Let $\mathcal{ISF}_n$ be the set of increasing spanning forests with n vertices. Define a cover relation on $\mathcal{ISF}_n$ by $F_1 \lessdot F_2$ if F_2 is obtained from F_1 by adding an edge between the smallest vertices of two trees in F_1.



$\mathcal{ISF}_3$

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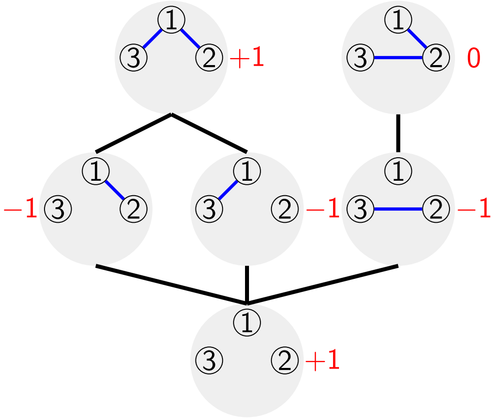


$w_2(\mathcal{ISF}_3) = 1$
$W_2(\mathcal{ISF}_3) = 2$

$w_1(\mathcal{ISF}_3) = -3$
$W_1(\mathcal{ISF}_3) = 3$

$w_0(\mathcal{ISF}_3) = 1$
$W_0(\mathcal{ISF}_3) = 1$

$\mathcal{ISF}_3$

Whitney Numbers of Π_n and $\mathcal{ISF}_n$

Compare the Whitney numbers of Π_3 and $\mathcal{ISF}_3$.

k	$w_k(\Pi_3)$	$W_k(\mathcal{ISF}_3)$	$w_k(\mathcal{ISF}_3)$	$W_k(\Pi_3)$
0	1	1	1	1
1	-3	3	-3	3
2	2	2	1	1

Whitney Duals

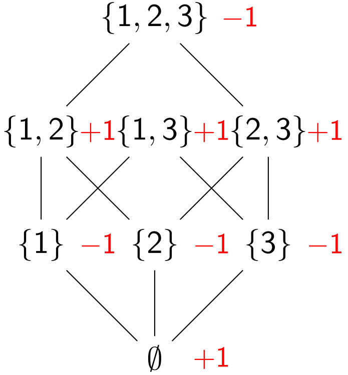
Let P and Q be graded posets both with $\hat{0}$s. We say that P and Q are **Whitney duals** if for all k,

$$|w_k(P)| = W_k(Q) \text{ and } |w_k(Q)| = W_k(P).$$

k	$\lvert w_k(\Pi_3)\rvert$	$W_k(\mathcal{ISF}_3)$	$\lvert w_k(\mathcal{ISF}_3)\rvert$	$W_k(\Pi_3)$
0	1	1	1	1
1	3	3	3	3
2	2	2	1	1

Thus the table above shows that Π_3 and $\mathcal{ISF}_3$ are Whitney duals. In fact, Π_n and $\mathcal{ISF}_n$ are Whitney duals for all n.

What posets have Whitney duals?



The boolean algebra is its own Whitney dual.

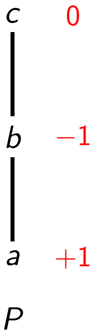
More generally, in any Eulerian poset,

$$\mu(x) = (-1)^{\rho(x)}$$

for all x and so $|w_k(P)| = W_k(P)$ for all k. Thus, all Eulerian posets have Whitney duals, namely themselves.

What posets have Whitney duals?

There are also posets which do not have Whitney duals.



$w_2(P) = 0$

$W_2(P) = 1$

$w_1(P) = -1$

$W_1(P) = 1$

$w_0(P) = 1$

$W_0(P) = 1$

If P had a Whitney dual, it would have to have no elements of rank two, but the absolute value of the sum of the Möbius values of the elements of rank two would need to be 1.

Constructing Whitney Duals

There are two main ingredients we use to construct Whitney duals.

1. Edge labelings
2. Quotient posets

Edge Labelings and Whitney Numbers

Definition

An **edge labeling**, λ, of poset of P is a labeling of the edges of the Hasse diagram of P where the set of labels is totally ordered. Given an edge labeling, we say a maximal chain $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k$ is **increasing** if

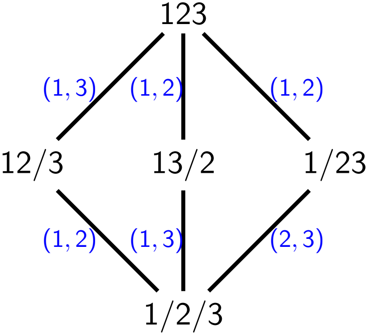
$$\lambda(x_1, x_2) < \lambda(x_2, x_3) < \cdots < \lambda(x_{k-1}, x_k).$$

Similarly, we say the chain is **ascent-free** if

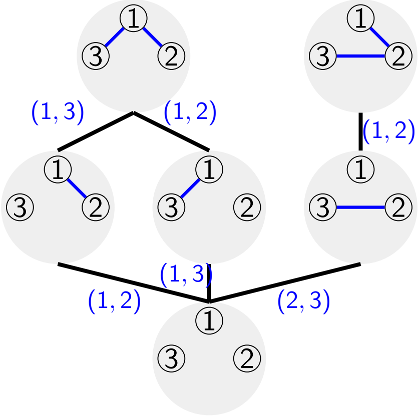
$$\lambda(x_1, x_2) \geq \lambda(x_2, x_3) \geq \cdots \geq \lambda(x_{k-1}, x_k).$$

We say λ is a an **ER-labeling** if every interval has a unique increasing maximal chain. Moreover, we say λ is an **ER*-labeling** if every interval has a unique ascent-free maximal chain.

Edge Labelings and Whitney Numbers



Π_3



$\mathcal{ISF}_3$

The labeling on Π_3 is an ER-labeling (unique maximal increasing chain in each interval) and the labeling on $\mathcal{ISF}_3$ is an ER*-labeling (unique maximal ascent-free chain in each interval).

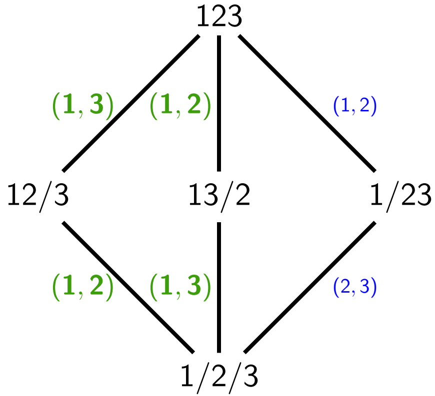
Edge Labelings and Whitney Numbers

Let P be a graded poset and suppose λ is an edge labeling. Then we have the following.

	$\lvert w_k(P) \rvert$	$W_k(P)$
λ is an ER-labeling	# (ascent-free sat. chains of length k starting at $\hat{0}$)	# (increasing sat. chains of length k starting at $\hat{0}$)
λ is an ER*-labeling	# (increasing sat. chains of length k starting at $\hat{0}$)	# (ascent-free sat. chains of length k starting at $\hat{0}$)

Edge Labelings and Whitney Numbers

Let λ be an ER-labeling of P. We say λ has the **rank two switching property** if for every interval $[x, y]$ with $\rho(y) - \rho(x) = 2$, if the unique increasing word is ab, then there is a unique maximal chain in $[x, y]$ labeled ba.



$\overline{\text{EW}}$-labelings

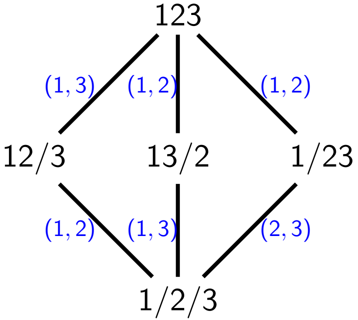
Definition

Let λ be an ER-labeling. We say λ is an **$\overline{\text{EW}}$-labeling** if the following hold.

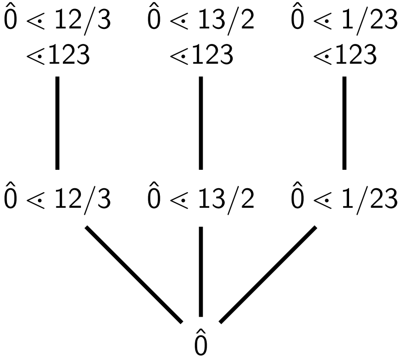
1. The set of labels is totally ordered.
2. λ has the rank two switching property.
3. For each interval $[x, y]$, if $\mathbf{c}$ and $\mathbf{c}'$ are distinct ascent-free maximal chains in $[x, y]$, then the sequence of labels of $\mathbf{c}$ and $\mathbf{c}'$ are different.

Chain Posets

Let P be a poset. The **chain poset** $C(P)$ is the set of saturated chains starting at $\hat{0}$ ordered by inclusion.



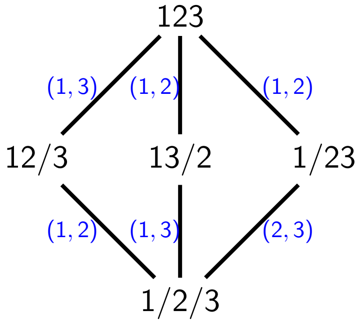
Π_3



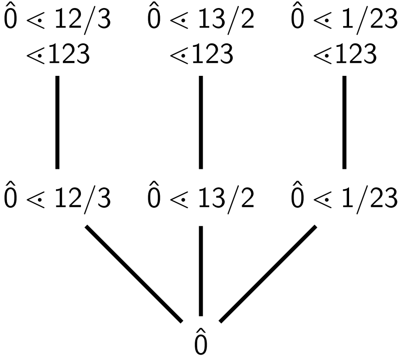
$C(\Pi_3)$

An Equivalence Relation

Suppose that λ is an $\overline{\text{EW}}$-labeling of P. We put an equivalence relation, $\sim_\lambda$, on the elements of the chain poset of P by saying $\mathbf{c} \sim_\lambda \mathbf{c}'$ if and only if $\mathbf{c}$ and $\mathbf{c}'$ terminate at the same element of P and the multiset of labels of $\mathbf{c}$ and $\mathbf{c}'$ are the same.



Π_3



$C(\Pi_3)$

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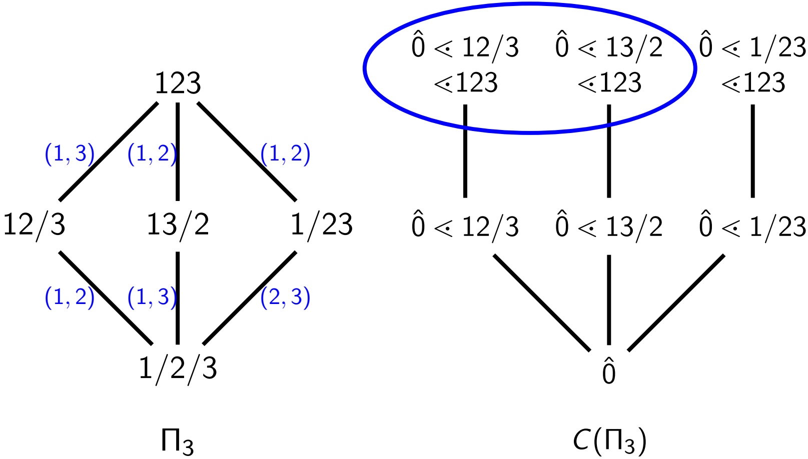

Quotient Posets

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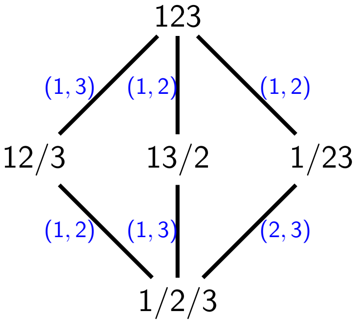
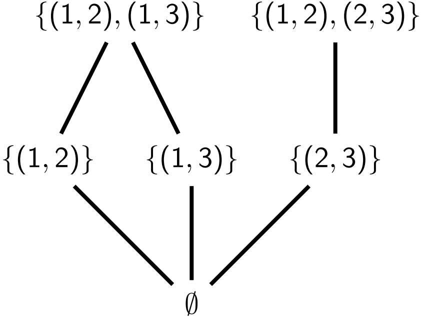


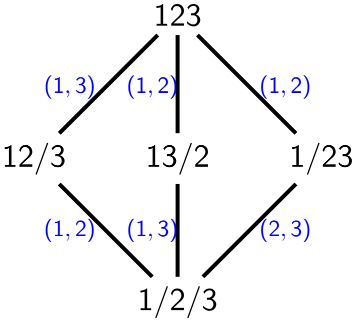
Π_3



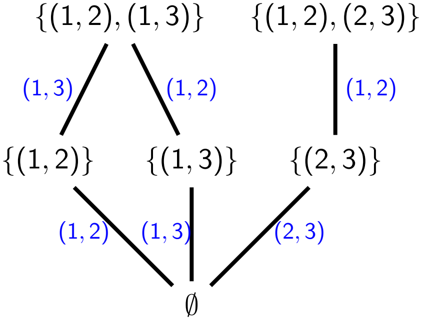
$C(\Pi_3)/\sim_\lambda$

Quotient Posets

By definition of $\sim_\lambda$, each element of a fixed equivalence class has the same multiset of labels. Define a labeling, λ^* on $C(P)/\sim_\lambda$ by setting $\lambda^*(X \lessdot Y)$ to be the unique label in Y not in X.



Π_3



$C(\Pi_3)/\sim_\lambda$

Quotient Posets

Proposition

Let λ be an $\overline{EW}$-labeling of P. Then λ^* is an ER^*-labeling of $C(P)/\sim_\lambda$.

Theorem

Let λ be an $\overline{EW}$-labeling of P. Then P and $C(P)/\sim_\lambda$ are Whitney duals.

Whitney Duals of Geometric Lattices

Definition (Björner, Stanley)

Let L be a geometric lattice and let $\preceq$ be a total order on the atoms of L. Define $\lambda(x \lessdot y) = a$ to be the smallest (with respect to $\preceq$) atom such that $a \vee x = y$.

Proposition

The previous labeling of geometric lattices is an $\overline{EW}$-labeling.

Corollary

Every geometric lattice has a Whitney dual.

Other Posets with Whitney Duals

- The noncrossing partition lattice
  - We use Stanley's parking function labeling. The Whitney dual is *noncrossing* increasing spanning forests ordering slightly differently than for increasing spanning forests.
- Weighted partition poset
- Weighted boolean poset

Questions

1. Can one classify which posets have Whitney duals?
2. Are there "nice" combinatorial descriptions of the Whitney duals we constructed using quotient posets?
3. What is the structure of the maximal intervals of the Whitney dual we constructed using quotient posets?
   - For geometric lattices, the intervals are supersolvable lattices.
   - For the noncrossing partition lattice the intervals are the orbits of the local S_n action on the maximal chains (i.e. the parking functions) described by Stanley.

THANK YOU!