Whitney Duals of Graded Partially Ordered Sets

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Partially Ordered Sets

A **partially ordered set (poset)** P is set together with a binary relation $\leq$ such that for all $x, y, z \in P$

1. $x \leq x$ (reflexive property)
2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetric property)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitive property)

The minimum element (if it exists) is denoted by $\hat{0}$. The maximum element (if it exists) is denoted by $\hat{1}$.

All the posets we consider w in this talk will be finite and have a $\hat{0}$, but necessarily a $\hat{1}$.

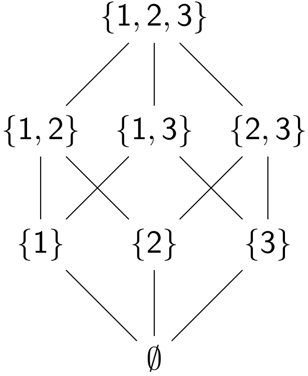
Examples of Posets

The **chain** C_n with n elements is the set $\{1, 2, \ldots, n\}$ with the normal ordering on $\mathbb{Z}$.

C_3

Examples of Posets

The **boolean algebra** B_n is the set of subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion.

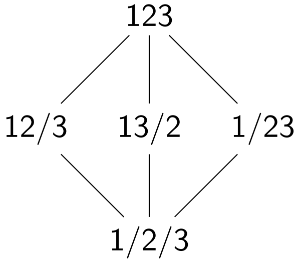


B_3

Examples of Posets

In addition to using $\leq$ to define the poset, we can also use the **cover relation**, $\lessdot$. Given a poset P, we say x is covered by y (or y covers x) and write $x \lessdot y$ if $x < y$ and there is no z such that $x < z < y$.

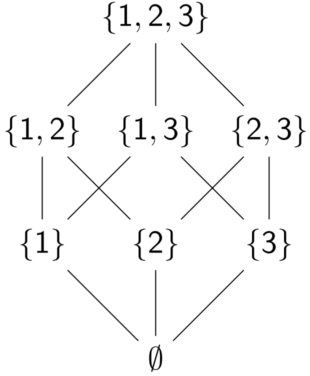
The **partition lattice** Π_n is the set of set partitions of $\{1, 2, \ldots, n\}$ with cover relation given by $\pi \lessdot \sigma$ if σ can be obtained from π by merging two blocks together.

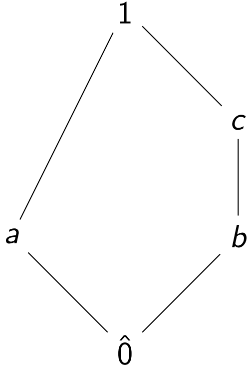


Π_3

Graded Posets

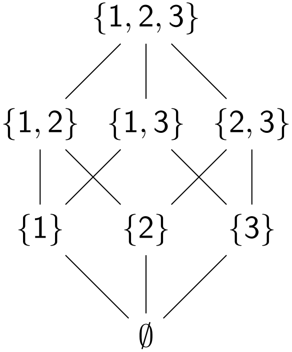
A **chain** in a poset is a totally ordered subset of the poset. We say a chain is **maximal** if is is not contained in any other chain. The **length** of a chain is one less the number of elements in the chain. If every maximal chain of a poset P has the same length, then we say P is **graded**.





Graded Posets

A **saturated chain** of a poset is a chain of the form $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k$. If P is a graded poset with a $\hat{0}$, then the **rank** function of P, denoted by ρ is defined setting $\rho(x)$ to be the length of a saturated chain from $\hat{0}$ to x.



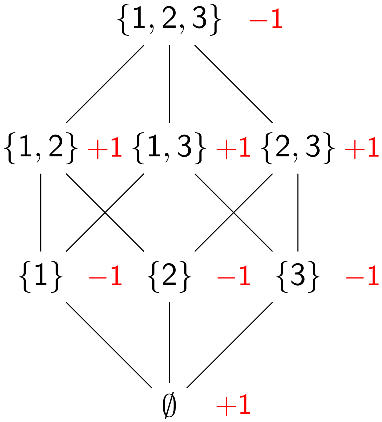
B_3

In B_n, $\rho(S) = |S|$.

The Möbius Function

The **(one–variable) Möbius function** $\mu : P \to \mathbb{Z}$ is defined by

$$\sum_{x \leq y} \mu(x) = \delta_{\hat{0},y}.$$



$$\mu(S) = (-1)^{|S|}$$

The Möbius Function

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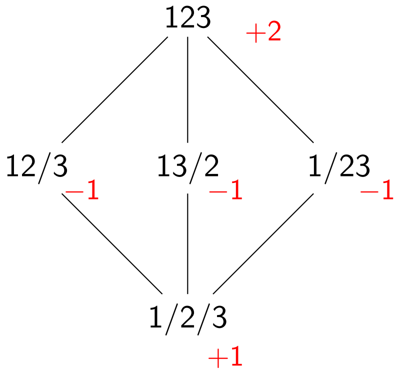


Π_3

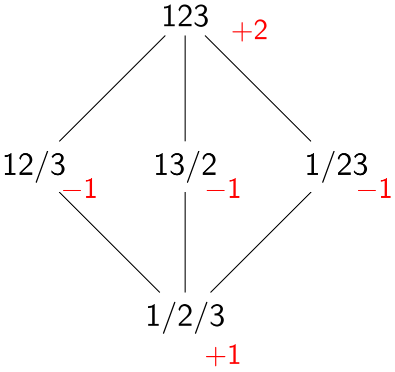
For $\pi = B_1/B_2/\ldots/B_k$,

$$\mu(\pi) = (-1)^{\rho(\pi)} \prod_{i=1}^{k} (|B_i| - 1)!$$

The Whitney Numbers

The k^{th}-**Whitney Number of the first kind** of P is defined by

$$w_k(P) = \sum_{\rho(x)=k} \mu(x).$$



Π_3

$w_2(\Pi_3) = 2$

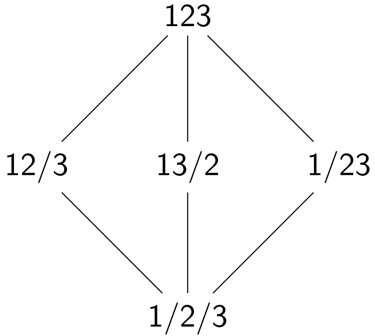
$w_1(\Pi_3) = -3$

$w_0(\Pi_3) = 1$

The Whitney Numbers

The k^{th}-**Whitney Number of the second kind** of P is defined by

$$W_k(P) = |\{x \in P \mid \rho(x) = k\}|.$$



Π_3

$W_2(\Pi_3) = 1$

$W_1(\Pi_3) = 3$

$W_0(\Pi_3) = 1$

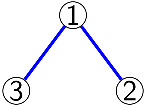
The Whitney Numbers

The Whitney numbers appear in several aspects of combinatorics. For example,

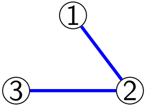
- The Stirling number of the first and second kind are the Whitney numbers of the first and second kind of the partition lattice Π_n.
- The coefficients of the chromatic polynomial of a graph are Whitney numbers of the first kind.
- (Zaslavsky) For a real affine hyperplane arrangement, the number of bounded regions is the sum of the Whitney numbers of the first kind of its intersection poset. The total number of regions is the sum of the absolute values of these numbers.

A Poset of Increasing Spanning Forests

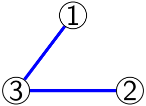
Let T be a tree with vertices labeled by distinct integers. We say T is an **increasing tree** if the labels along any path starting at the smallest vertex increase.



Increasing



Increasing

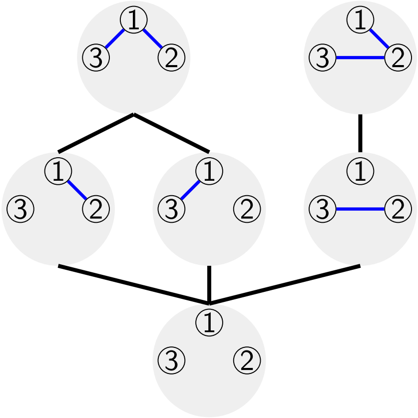


Not Increasing

A **increasing spanning forest** is a forest with vertex set $\{1, 2, \ldots, n\}$ such that each connected component is an increasing tree.

A Poset of Increasing Spanning Forests

Let $\mathcal{ISF}_n$ be the set of increasing spanning forests with n vertices. Define a cover relation on $\mathcal{ISF}_n$ by $F_1 \lessdot F_2$ if F_2 is obtained from F_1 by adding an edge between the smallest vertices of two trees in F_1.



$\mathcal{ISF}_3$

A Poset of Increasing Spanning Forests

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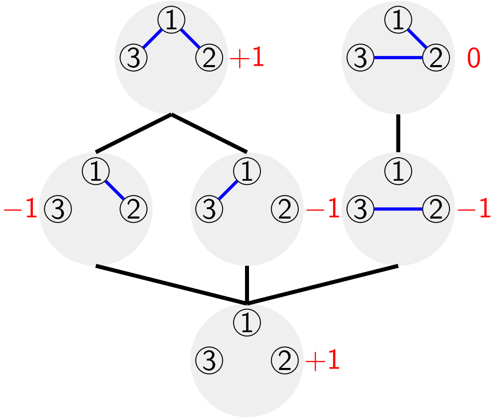


$$w_2(\mathcal{ISF}_3) = 1$$
$$W_2(\mathcal{ISF}_3) = 2$$

$$w_1(\mathcal{ISF}_3) = -3$$
$$W_1(\mathcal{ISF}_3) = 3$$

$$w_0(\mathcal{ISF}_3) = 1$$
$$W_0(\mathcal{ISF}_3) = 1$$

$\mathcal{ISF}_3$

Whitney Numbers of Π_3 and $\mathcal{ISF}_3$

Compare the Whitney numbers of Π_3 and $\mathcal{ISF}_3$.

k	$w_k(\Pi_3)$	$W_k(\mathcal{ISF}_3)$	$w_k(\mathcal{ISF}_3)$	$W_k(\Pi_3)$
0	1	1	1	1
1	-3	3	-3	3
2	2	2	1	1

Whitney Duals

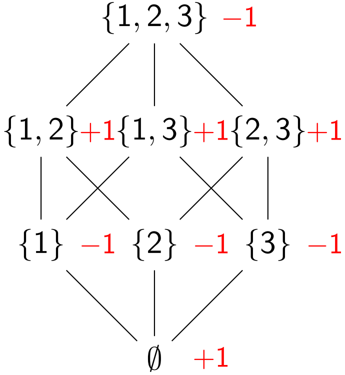
Let P and Q be graded posets both with $\hat{0}$s. We say that P and Q are **Whitney duals** if for all k,

$$|w_k(P)| = W_k(Q) \text{ and } |w_k(Q)| = W_k(P).$$

k	$\lvert w_k(\Pi_3)\rvert$	$W_k(\mathcal{ISF}_3)$	$\lvert w_k(\mathcal{ISF}_3)\rvert$	$W_k(\Pi_3)$
0	1	1	1	1
1	3	3	3	3
2	2	2	1	1

Thus the table above shows that Π_3 and $\mathcal{ISF}_3$ are Whitney duals. In fact, Π_n and $\mathcal{ISF}_n$ are Whitney duals for all n.

What posets have Whitney duals?



The boolean algebra is its own Whitney dual.

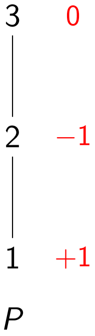
More generally, in any Eulerian poset,

$$\mu(x) = (-1)^{\rho(x)}$$

for all x and so $|w_k(P)| = W_k(P)$ for all k. Thus, all Eulerian posets have Whitney duals, namely themselves.

What posets have Whitney duals?

There are also posets which do not have Whitney duals.



$w_2(P) = 0$

$W_2(P) = 1$

$w_1(P) = -1$

$W_1(P) = 1$

$w_0(P) = 1$

$W_0(P) = 1$

If P had a Whitney dual, it would have to have no elements of rank two, but the absolute value of the sum of the Möbius values of the elements of rank two would need to be 1.

Constructing Whitney Duals

There are two main ingredients we use to construct Whitney duals.

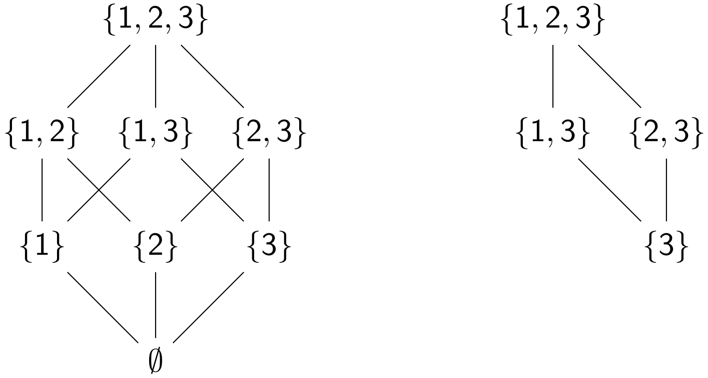
1. Edge labelings
2. Quotient posets

Intervals

Let P be a poset and let $x, y \in P$. The **(closed) interval** $[x, y]$ is given by

$$[x, y] = \{z \mid x \leq z \leq y\}.$$

For example, in B_3 we have



B_3 $[\{3\}, \{1, 2, 3\}]$

Edge Labelings and Whitney Numbers

Definition

An **edge labeling**, λ, of poset of P is a labeling of the edges of the Hasse diagram of P where the set of labels is partially ordered. Given an edge labeling, we say a saturated chain $x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k$ is **increasing** if

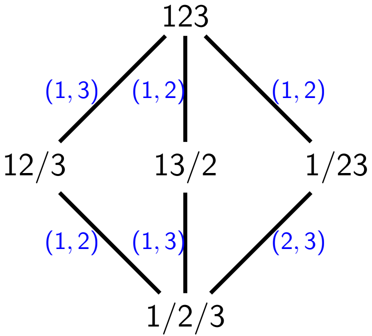
$$\lambda(x_1, x_2) < \lambda(x_2, x_3) < \cdots < \lambda(x_{k-1}, x_k).$$

Similarly, we say the chain is **ascent-free** if

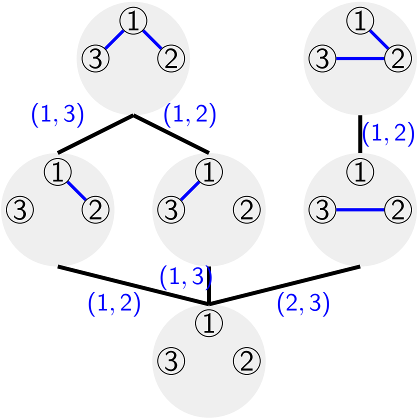
$$\lambda(x_1, x_2) \not< \lambda(x_2, x_3) \not< \cdots \not< \lambda(x_{k-1}, x_k).$$

We say λ is a an **ER-labeling** if every interval has a unique increasing maximal chain. Moreover, we say λ is an **ER*-labeling** if every interval has a unique ascent-free maximal chain.

Edge Labelings and Whitney Numbers



Π_3



$\mathcal{ISF}_3$

The labeling on Π_3 is an ER-labeling (unique maximal increasing chain in each interval) and the labeling on $\mathcal{ISF}_3$ is an ER*-labeling (unique maximal ascent-free chain in each interval).

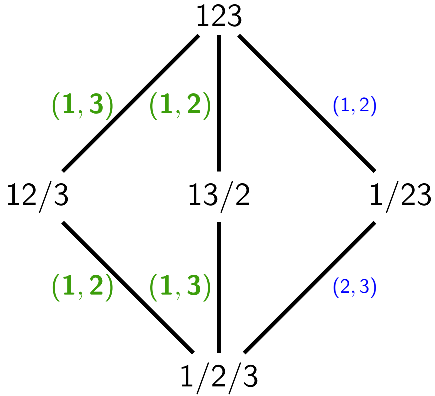
Edge Labelings and Whitney Numbers

Let P be a graded poset and suppose λ is an edge labeling. Then we have the following.

	$\lvert w_k(P) \rvert$	$W_k(P)$
λ is an ER-labeling	# (ascent-free sat. chains of length k starting at $\hat{0}$)	# (increasing sat. chains of length k starting at $\hat{0}$)
λ is an ER*-labeling	# (increasing sat. chains of length k starting at $\hat{0}$)	# (ascent-free sat. chains of length k starting at $\hat{0}$)

Edge Labelings and Whitney Numbers

Let λ be an ER-labeling of P. We say λ has the **rank two switching property** if for every interval $[x, y]$ with $\rho(y) - \rho(x) = 2$, if the unique increasing word is ab, then there is a unique maximal chain in $[x, y]$ labeled ba.



EW-labelings

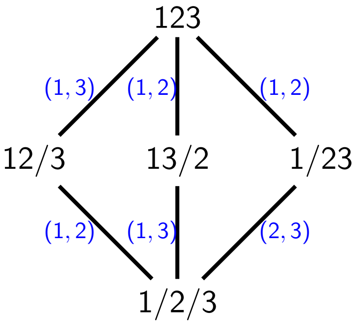
Definition

Let λ be an ER-labeling. We say λ is an **EW-labeling** if the following hold.

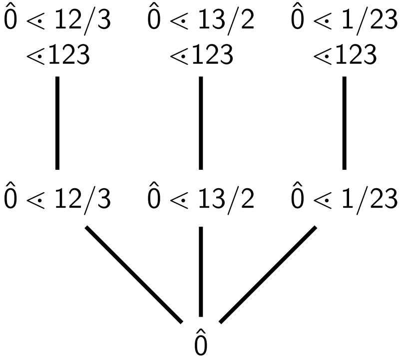
1. λ has the rank two switching property.
2. For each interval $[x, y]$, if $\mathbf{c}$ and $\mathbf{c}'$ are distinct ascent-free maximal chains in $[x, y]$, then the sequence of labels of $\mathbf{c}$ and $\mathbf{c}'$ are different.

Chain Posets

Let P be a poset. The **chain poset** $C(P)$ is the set of saturated chains starting at $\hat{0}$ ordered by inclusion.



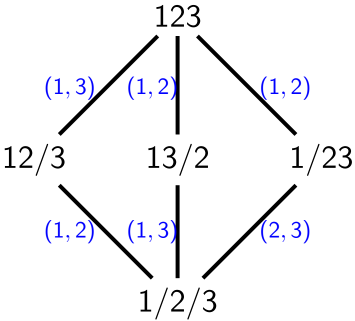
Π_3



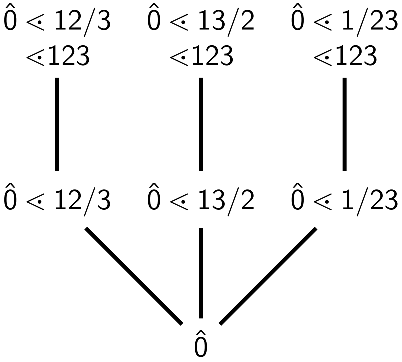
$C(\Pi_3)$

An Equivalence Relation

Suppose that λ is an EW-labeling of P. We put an equivalence relation, $\sim_\lambda$, on the elements of the chain poset of P by saying $\mathbf{c} \sim_\lambda \mathbf{c}'$ if and only if $\mathbf{c}$ and $\mathbf{c}'$ are related by a sequences of rank two switches.



Π_3



$C(\Pi_3)$

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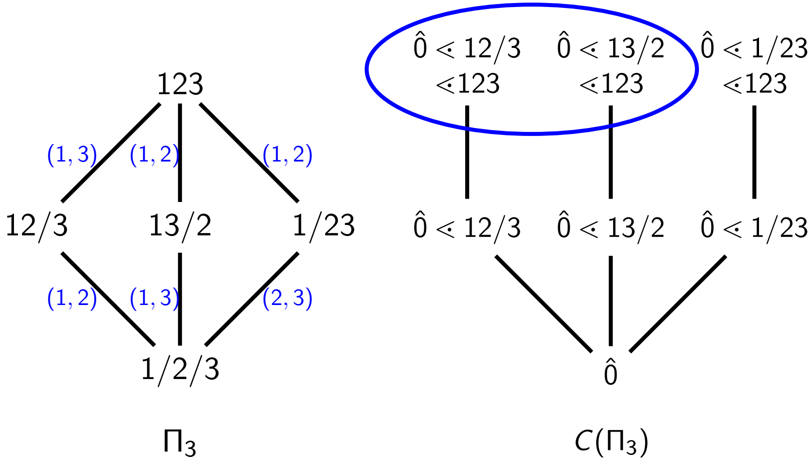
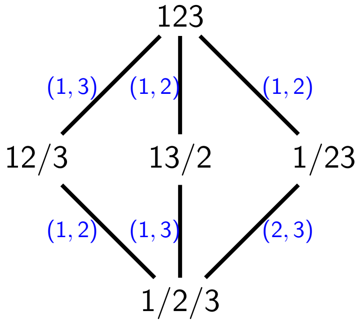


Quotient Posets

Suppose that λ is an EW-labeling of P. We put an equivalence relation, $\sim_\lambda$, on the elements of the chain poset of P by saying $\mathbf{c} \sim_\lambda \mathbf{c}'$ if and only if $\mathbf{c}$ and $\mathbf{c}'$ are related by a sequences of rank two switches.



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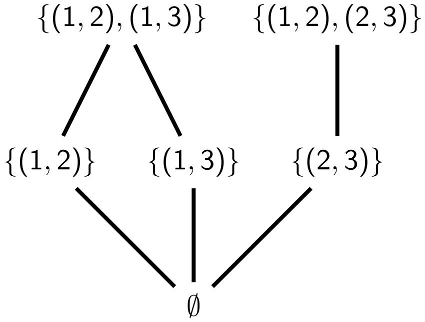
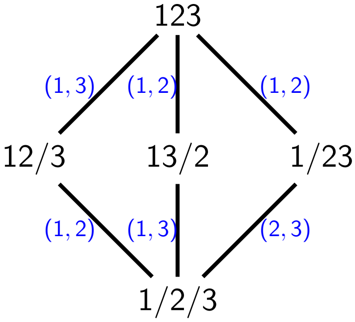


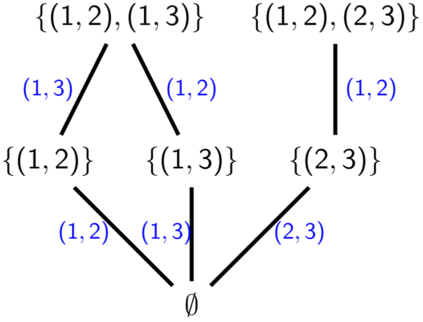
$C(\Pi_3)/\sim_\lambda$

Quotient Posets

By definition of $\sim_\lambda$, each element of a fixed equivalence class has the same multiset of labels. Define a labeling, λ^* on $C(P)/\sim_\lambda$ by setting $\lambda^*(X \lessdot Y)$ to be the unique label in Y not in X.



Π_3



$C(\Pi_3)/\sim_\lambda$

Quotient Posets

Proposition

Let λ be an EW-labeling of P. Then λ^* is an ER-labeling of $C(P)/\sim_\lambda$.*

Proposition

There is a label preserving bijection between saturated chains of length k starting at $\hat{0}$ in P and saturated chains of length k starting at $\hat{0}$ in $C(P)/\sim_\lambda$.

Quotient Posets

Proposition

Let λ be an EW-labeling of P. Then λ^* is an ER^*-labeling of $C(P)/\sim_\lambda$.

Proposition

There is a label preserving bijection between saturated chains of length k starting at $\hat{0}$ in P and saturated chains of length k starting at $\hat{0}$ in $C(P)/\sim_\lambda$.

	$\lvert w_k(P) \rvert$	$W_k(P)$
λ is an ER-labeling	# (ascent-free sat. chains of length k starting at $\hat{0}$)	# (increasing sat. chains of length k starting at $\hat{0}$)
λ is an ER*-labeling	# (increasing sat. chains of length k starting at $\hat{0}$)	# (ascent-free sat. chains of length k starting at $\hat{0}$)

Theorem

Let λ be an EW-labeling of P. Then P and $C(P)/\sim_\lambda$ are Whitney duals.

Examples: Geometric Lattices

Definition (Björner, Stanley)

Let L be a geometric lattice and let $\preceq$ be a total order on the atoms of L. Define $\lambda(x \lessdot y) = a$ to be the smallest (with respect to $\preceq$) atom such that $a \vee x = y$.

Proposition

The previous labeling of geometric lattices is an EW-labeling.

Corollary

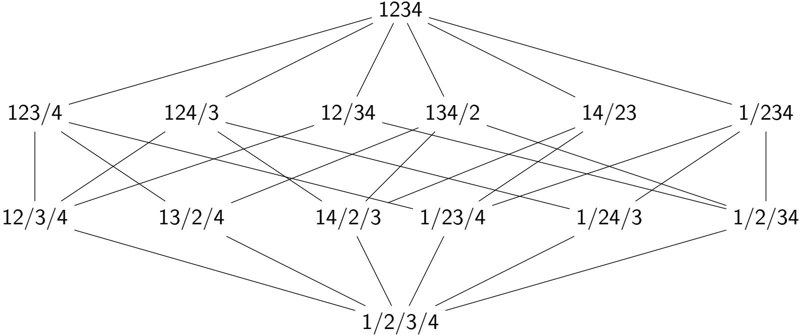
Every geometric lattice has a Whitney dual.

Examples: The Noncrossing Partition Lattice

We say a partition $B_1/B_2/\cdots/B_k$ is **crossing** if there exists $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$ and $a < b < c < d$. If partition is not crossing, we call it a **noncrossing partition**. For example, the partition 13567/24 is crossing whereas 1567/234 is noncrossing.

Examples: The Noncrossing Partition Lattice

The set of noncrossing partitions of $\{1, 2, \ldots, n\}$ forms a lattice with cover relation $\pi \lessdot \sigma$ if σ can be obtained from π by merging two blocks together. The lattice is called the **noncrossing partition lattice** and is denoted by NC_n.



NC_4

Examples: The Noncrossing Partition Lattice

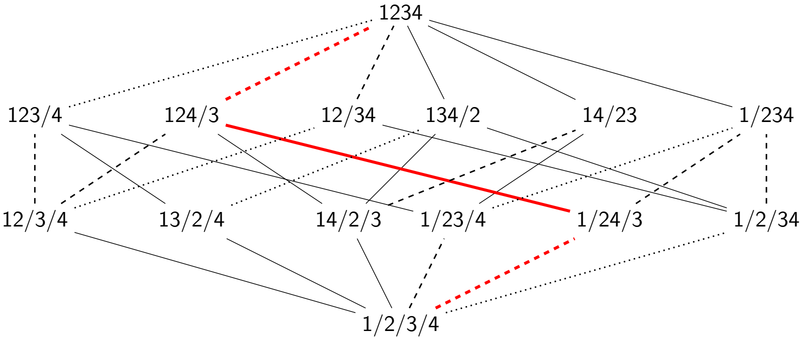
A **parking function** is sequence $(a_1, a_2, \ldots, a_n)$ of positive integers such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $(a_1, a_2, \ldots, a_n)$, then $b_i \leq i$ for all i.

For example, $(2, 1, 2)$ is a parking function whereas $(2, 2, 2)$ is not.

Stanley found an edge labeling of NC_n where the labels along maximal chains are exactly the parking functions.

Examples: The Noncrossing Partition Lattice

The labeling is given by $\lambda(\pi \lessdot \sigma) = \max\{a \in B_i \mid a < \min B_j\}$ where σ is obtained from π by merging B_i and B_j with $\min B_i < \min B_j$.



Solid lines correspond to 1

Dashed lines correspond to 2

Dotted lines correspond to 3

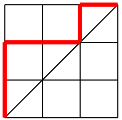
Examples: The Noncrossing Partition Lattice

Theorem

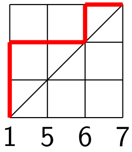
Stanley's parking function labeling of NC_n is an EW-labeling. Consequently, NC_n has a Whitney dual.

Examples: The Noncrossing Partition Lattice

A **Dyck path** is a northeast lattice path from $(0, 0)$ to (n, n) which stays weakly above the line $y = x$.

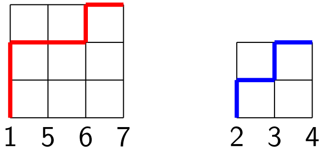


We will label the bottom of the diagram with a set of positive integers so that the sequence is increasing. For example, if we choose the set $\{1, 5, 6, 7\}$ we have

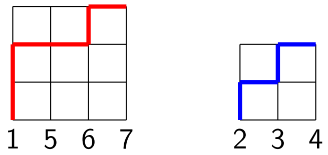


Examples: The Noncrossing Partition Lattice

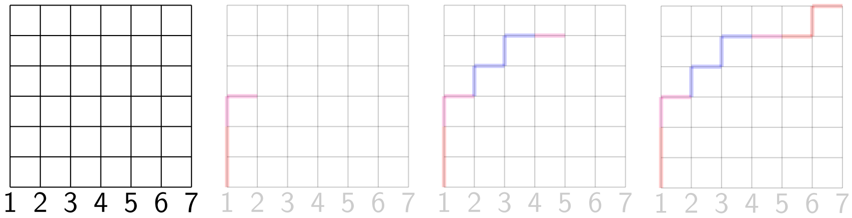
Suppose that we have two Dyck paths labeled by set S and T such that S/T is a noncrossing partition. We will describe how to merge together the Dyck paths with an example.



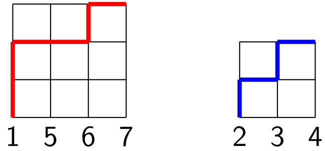
Examples: The Noncrossing Partition Lattice



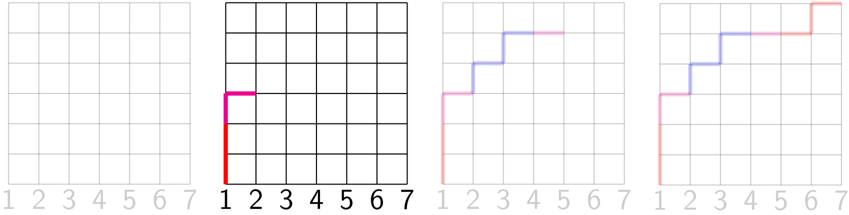
The first step is to create a labeled grid.



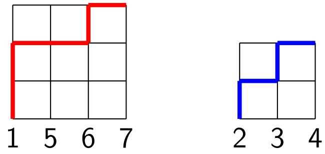
Examples: The Noncrossing Partition Lattice



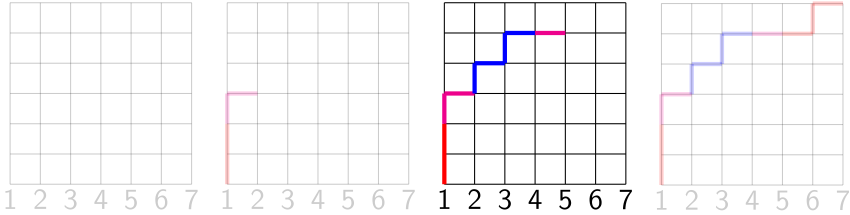
Next, find the largest value of the label along the first Dyck path that is smaller than all other labels in the second Dyck path. In this case, it is 1. Now take the first Dyck path and add a new north step above 1 followed by an east step.



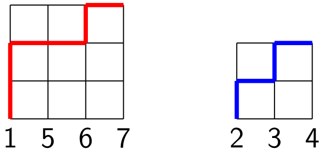
Examples: The Noncrossing Partition Lattice



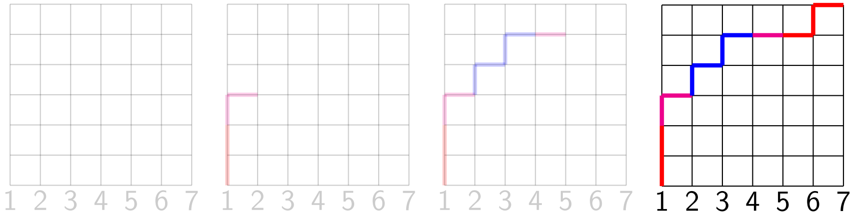
Now shift up the the second Dyck path so that it meets the Dyck path we have just drawn. Then glue it in and add an east step.



Examples: The Noncrossing Partition Lattice

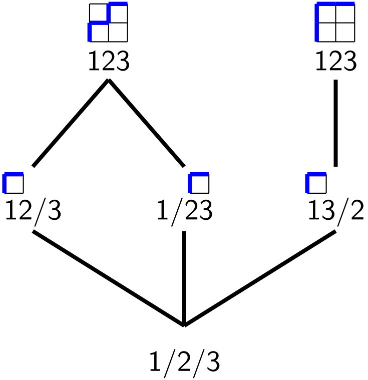


Finally, shift the remaining part of the first Dyck path up and glue it on.



Examples: The Noncrossing Partition Lattice

Now consider all noncrossing partitions of $[n]$ such that each block has a Dyck path above it. Put a cover relation on these elements so that going up in the cover relation is done by merging two Dyck paths together.



A 0-Hecke Algebra Action

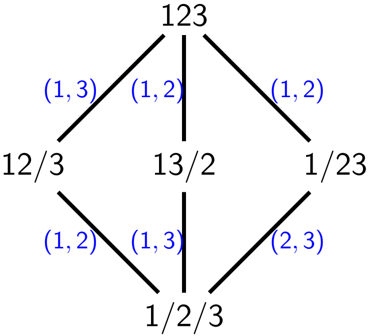
Suppose that P is a poset of rank n. Let $\mathcal{M}(P)$ be the set of maximal chains. If P has an EW-labeling, then for each $1 \leq i \leq n-1$, we can define maps $U_i : \mathcal{M}(P) \to \mathcal{M}(P)$ by sending $\mathbf{c}$

$$U_i(\mathbf{c}) = \begin{cases} \mathbf{c}' & \text{if } \mathbf{c} \text{ has an ascent at position } i, \\ \mathbf{c} & \text{otherwise} \end{cases}$$

where $\mathbf{c}'$ is the unique maximal chain obtained by applying a rank 2 move at rank i.

A 0-Hecke Algebra Action

Example



Π_3

$$U_1(1/2/3 \lessdot 12/3 \lessdot 123) = 1/2/3 \lessdot 13/2 \lessdot 123$$
$$U_1(1/2/3 \lessdot 13/2 \lessdot 123) = 1/2/3 \lessdot 13/2 \lessdot 123$$
$$U_1(1/2/3 \lessdot 1/23 \lessdot 123) = 1/2/3 \lessdot 1/23 \lessdot 123$$

A 0-Hecke Algebra Action

It turns out when the labeling on the poset is an EW-labeling, the U_i's have the following properties.

1. $U_i^2 = U_i$ for all i
2. $U_i U_j = U_j U_i$ for all $|i - j| > 1$.
3. $U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}$ for all i

Thus, the U_i's generate the 0-Hecke algebra, $H_n(0)$. We can linearly extend the maps U_i to $\mathbb{C}\mathcal{M}(P)$ to produce a local action on this vector space.

Theorem

The quasisymmetric characteristic of this action on poset P with an EW-labeling is Ehrenborg's flag quasisymmetric function of P.

Questions

1. Can we develop a construction to "go the other way"?
2. Can one classify which posets have Whitney duals?
3. Are there "nice" combinatorial descriptions of the Whitney duals we constructed using quotient posets?
4. What is the structure of the maximal intervals of the Whitney dual we constructed using quotient posets?

THANK YOU!