Braid cones and the Gorenstein property

Josh Hallam
Loyola Marymount University

John Machacek
University of Oregon

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Overview

Connected Posets $\rightarrow$ Braid Cones $\rightarrow$ Toric Varieties

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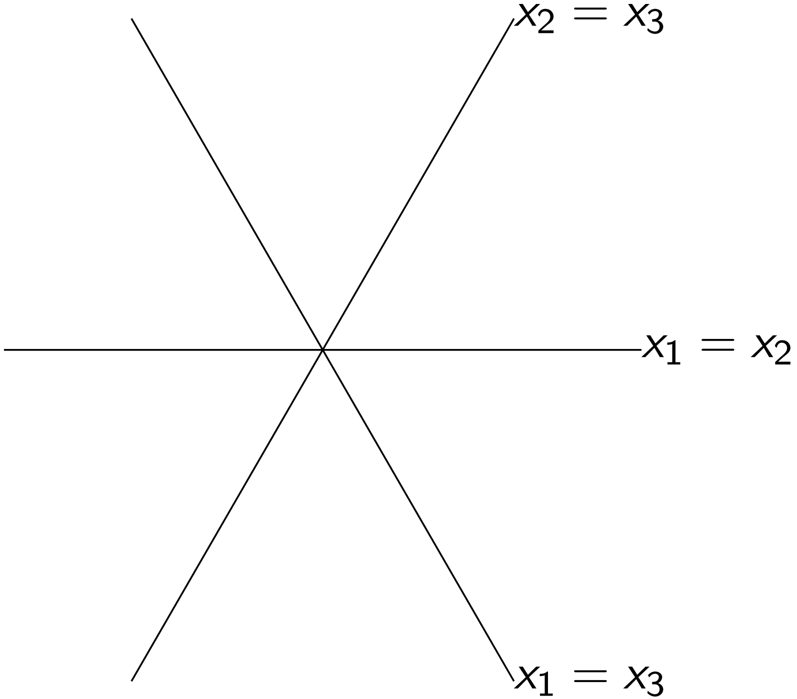
Connected Posets $\rightarrow$ **Braid Cones** $\rightarrow$ **Toric Varieties**

Combinatorial properties of the poset translate to algebraic properties of the toric variety and vice versa.

Braid Cones

The **braid arrangement** in $\mathbb{R}^n$ is the collection of hyperplanes $\{x_i = x_j \mid 1 \leq i < j \leq n\}$.



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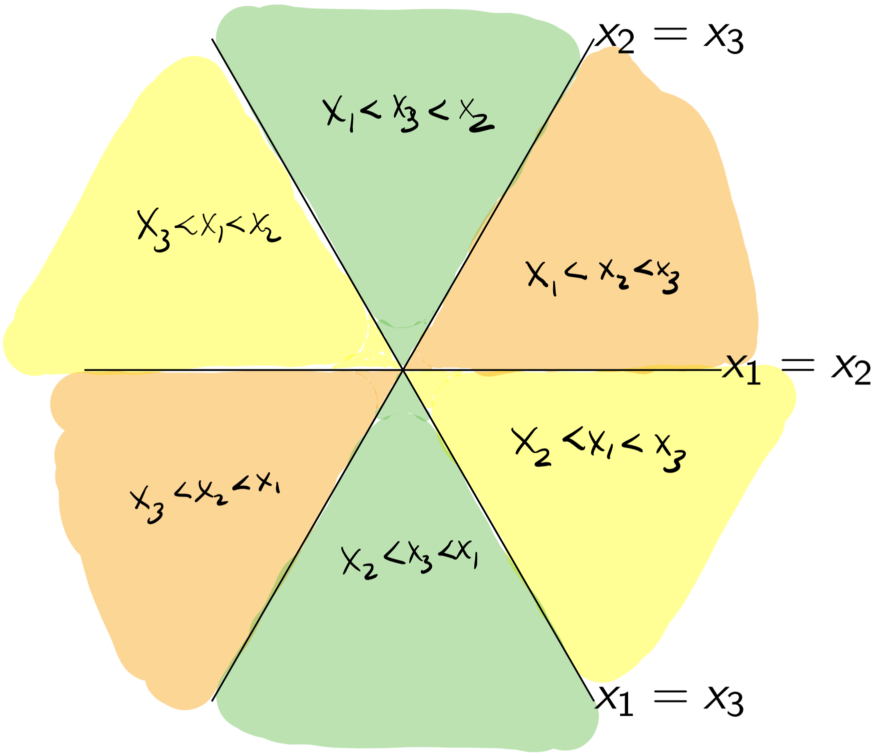
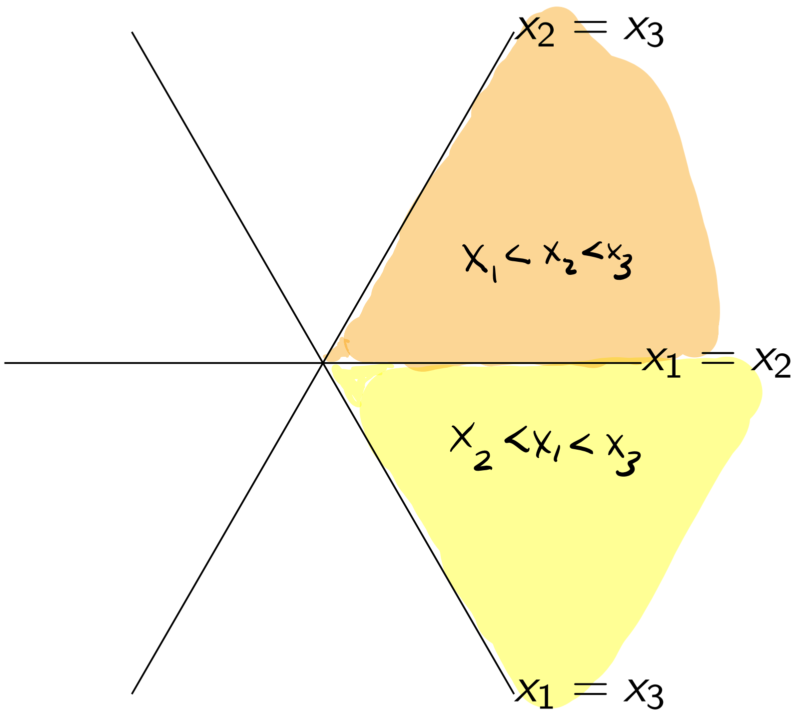
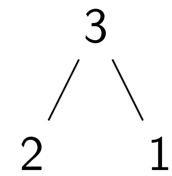


The hyperplanes break $\mathbb{R}^n$ into regions. These regions are indexed by linear orders on $\{x_1, x_2, \ldots, x_n\}$.

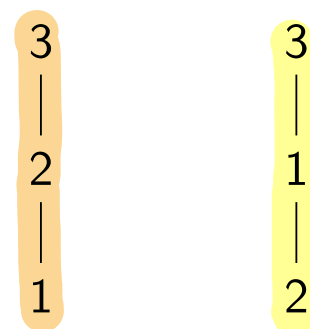
Braid Cones

Given a poset P on $[n]$. Each linear extension of P gives a region in the braid arrangement. The braid cone associated to P is the union of these regions over all linear extension of P.





P



Linear Extensions

The Algebraic Properties of Braid Cones

Let P be a connected poset on $[n]$, we will use σ_P to denote the braid cone. We will also use U_P to denote the toric variety associated to σ_P.

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U_P is smooth if and only if the Hasse diagram of P is a tree.

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Theorem (Postnikov, Reiner, Williams 2008)

U_P is smooth if and only if the Hasse diagram of P is a tree.

It is also known that U_P is Cohen-Macaulay for all posets P.

The Gorenstein Property

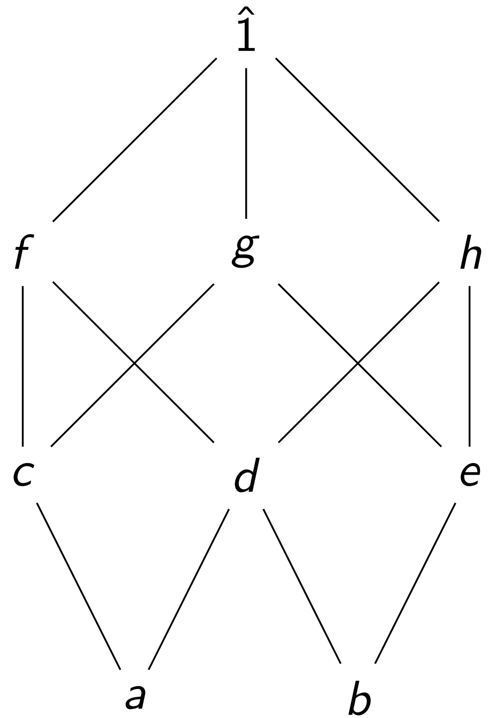
Sitting properly between smooth and Cohen-Macaulay varieties is the class of Gorenstein varieties. One can think of these as varieties that are "not too singular".

The Gorenstein property has been studied for other varieties arising in combinatorics.

- Schubert Varieties (Woo and Yong 2006)
- Varieties arising from the base polytope of graphical matroids (Hibi, Lasoń, Matsuda, Michałek, and Vodička 2021)

The Gorenstein Property for Braid Cones

Let P be a poset. We say $S \subseteq P$ is an **downset** if $x \in S$ and $y < x$ implies that $y \in P$. Note that a downset is generated by its maximal elements.



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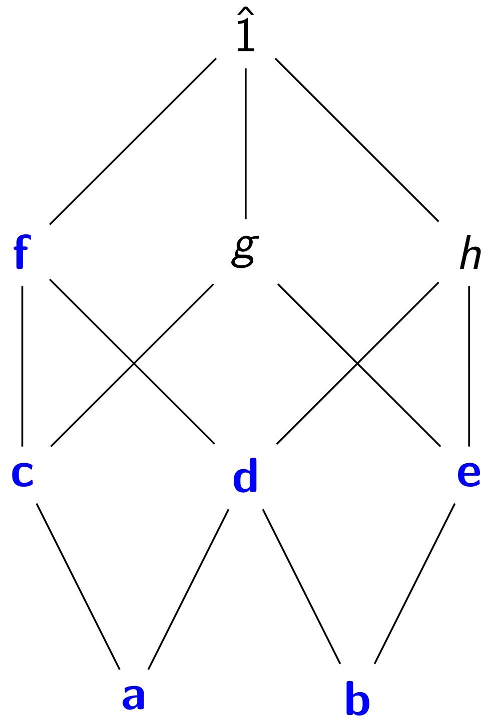

$\{a, b, c, d, e, f\}$ is a downset generated by f and e.

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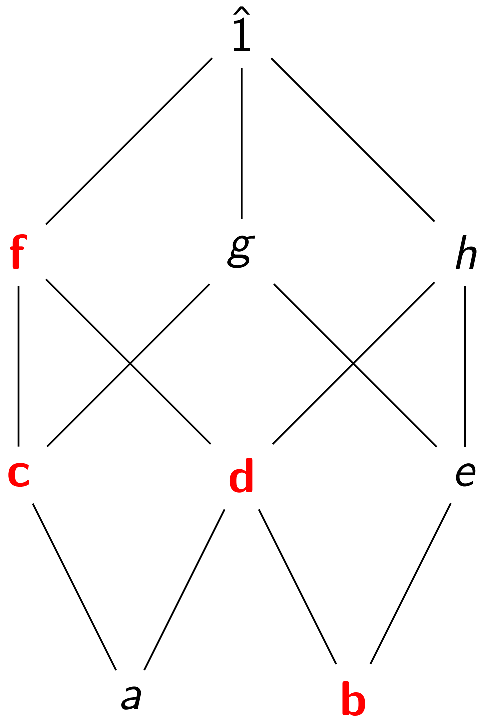


$\{b, c, d, f\}$ is not a downset.

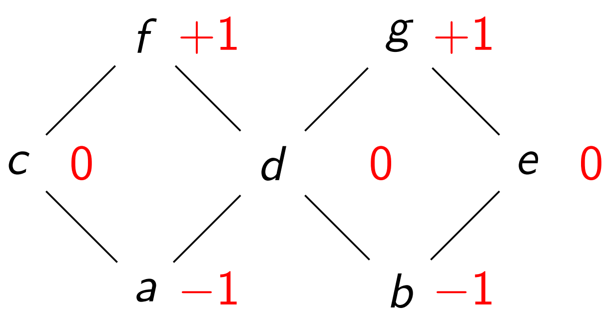
The Gorenstein Property for Braid Cones

We say a vertex labeling $\phi : P \to \mathbb{Z}$ is a **Gorenstein labeling** if

$$\sum_{i \in P} \phi(i) = 0$$

and for all downsets S such that S and $P \setminus S$ are both connected and proper

$$\sum_{i \in S} \phi(i) = -1.$$



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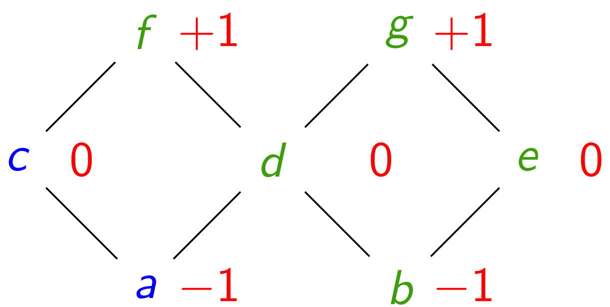

$\{a, c\}$ and its complement are both connected.

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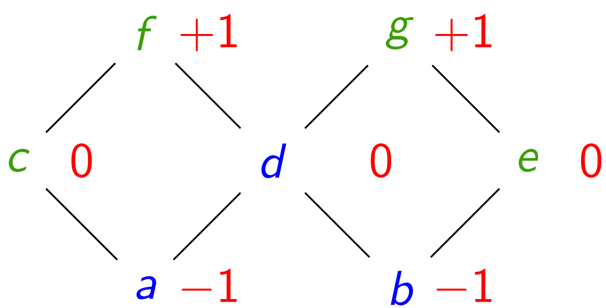

$\{a, b, d\}$ is connected, but its complement $\{c, e, f, g\}$ is not.

The Gorenstein Property for Braid Cones

Theorem

U_P is Gorenstein if and only if P has a Gorenstein labeling.

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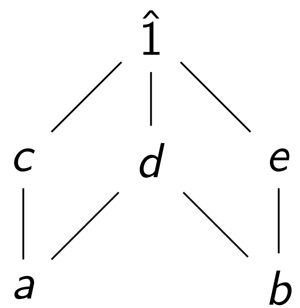
Examples of posets that have Gorenstein labelings include

- Posets with a maximum and minimum.
- Posets of rank 1, whose Hasse diagram is $K_{n,n}$.
- Posets obtained by gluing two posets with Gorenstein labelings along a vertex.
- Posets whose underlying Hasse diagram is a tree.
- Posets whose order dual has a Gorenstein labeling.

Posets with a maximum or a minimum

- Now lets assume that P has a maximum or a minimum. Since a poset has Gorenstein labeling if and only if its order dual has one, we can assume P has a maximum, $\hat{1}$.
- Since the complement of a proper downset contains $\hat{1}$, the complement will be connected.
- Since downsets generated by single elements are always connected, a Gorenstein labeling is completely determined by such downsets.



The Möbius Function

Let P be a poset with a minimum element $\hat{0}$. The **(one variable) Möbius function** is defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{otherwise} \end{cases}$$

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Proposition

Let P be a poset with $\hat{1}$. If P has a Gorenstein labeling ϕ, then

$$\phi(x) = \begin{cases} \mu_{\hat{P}}(x) & \text{if } x \neq \hat{1}, \\ \mu_{\hat{P}}(x) + 1 & \text{if } x = \hat{1}. \end{cases}$$

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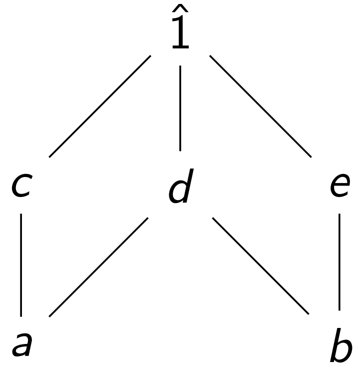

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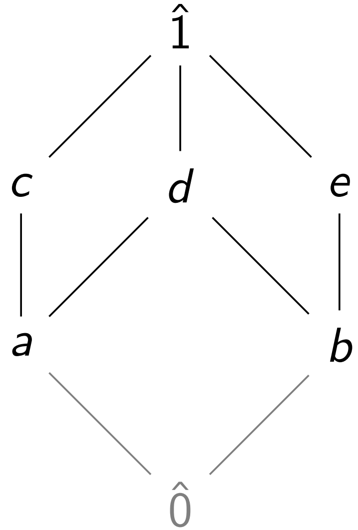

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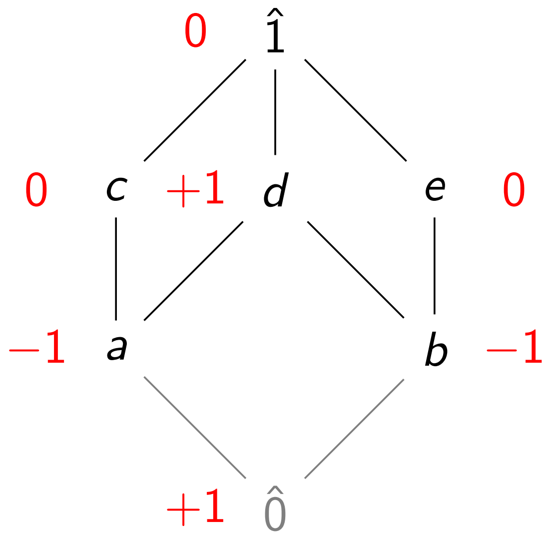

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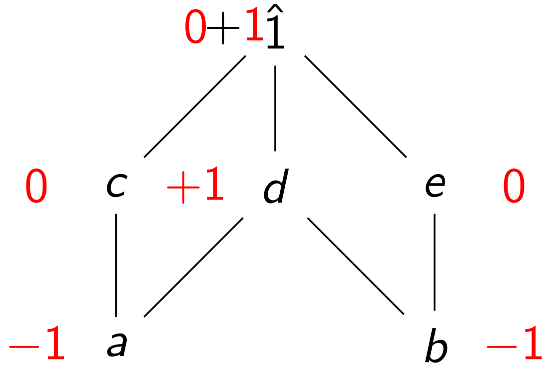
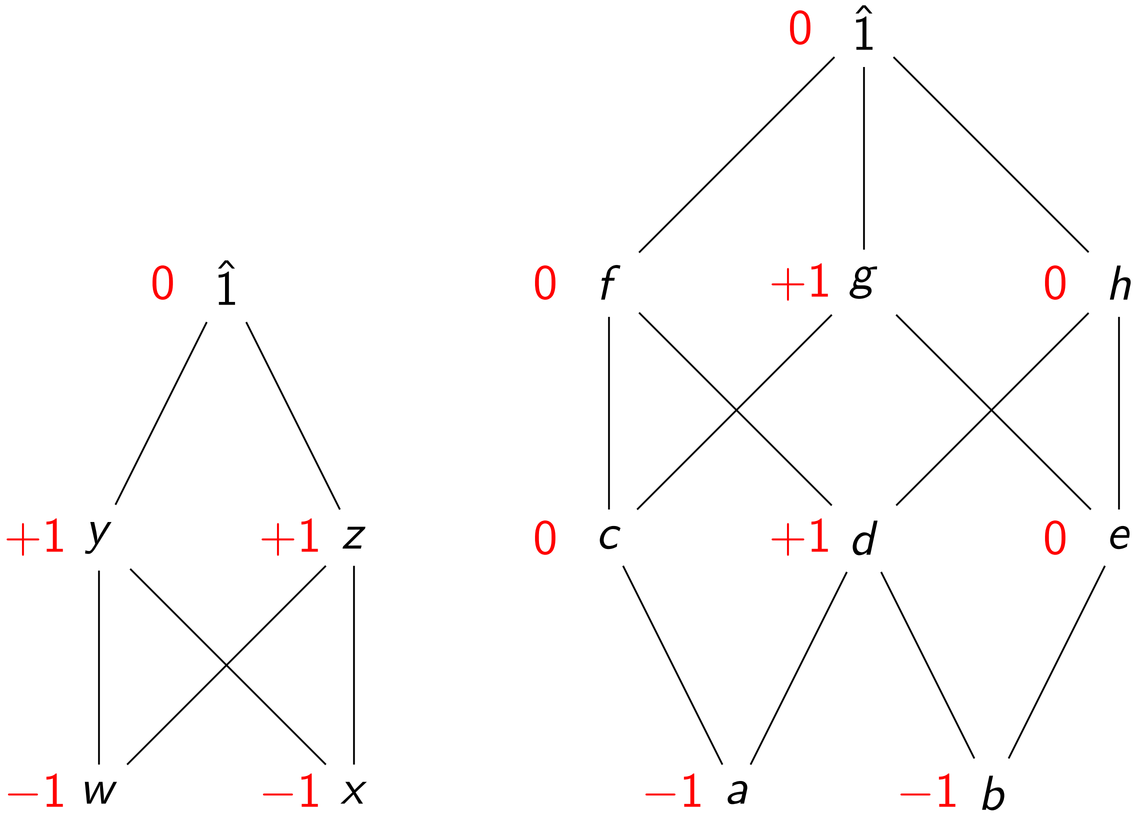


Gorenstein Labeling

Posets with a Maximum

Using the Möbius function, we get the the condition for a Gorenstein labeling when we look at downsets generated by a single element. But what about other downsets?



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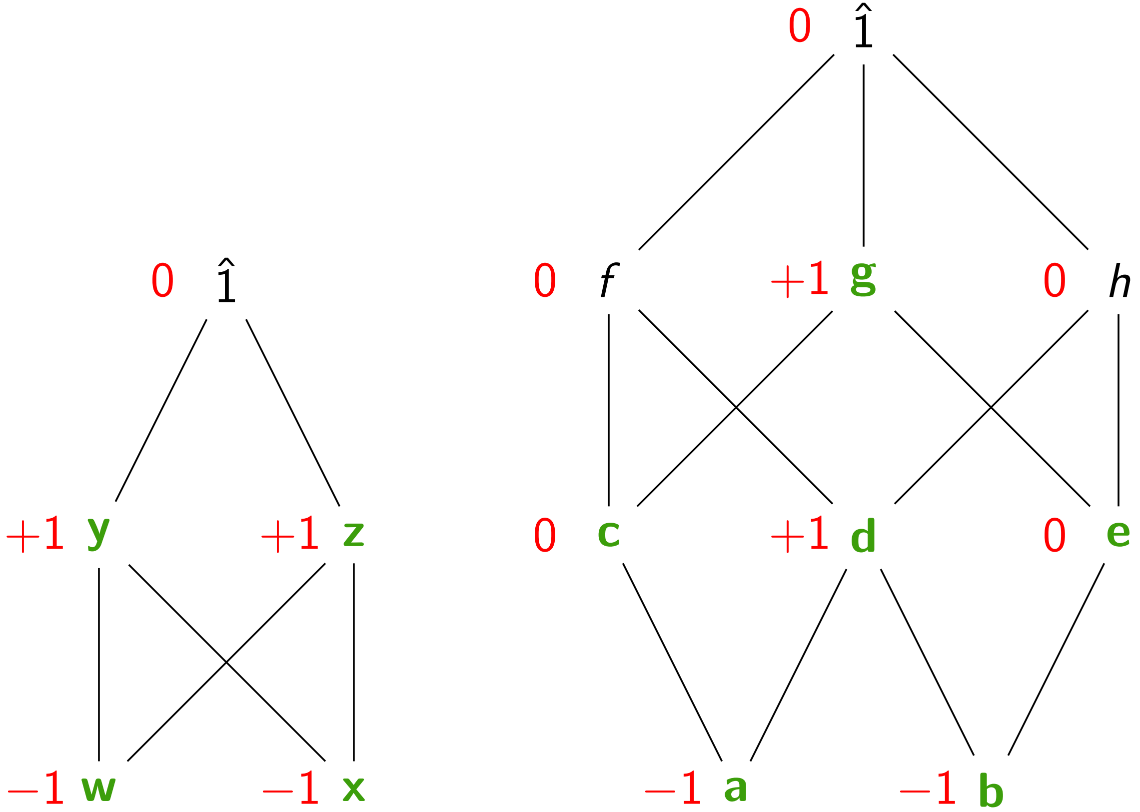
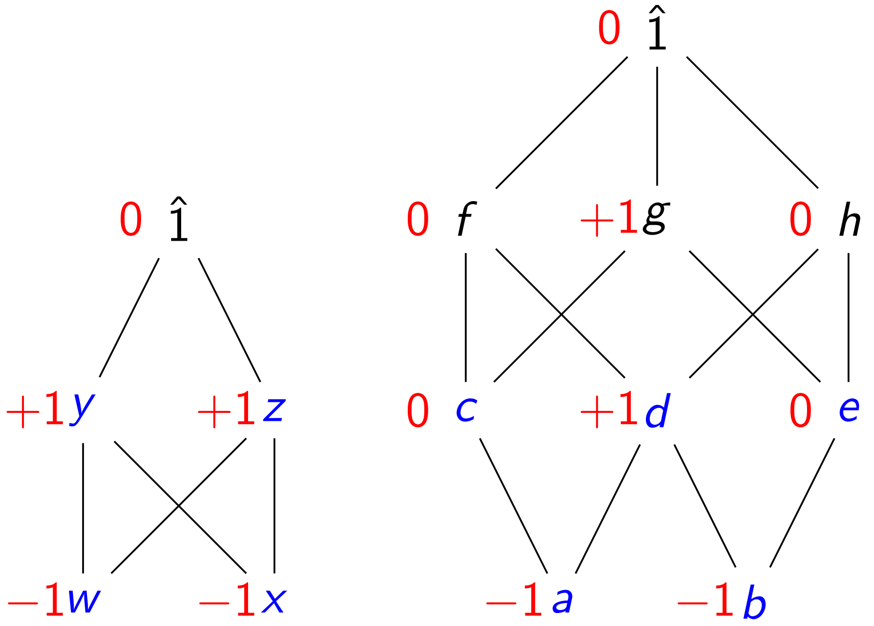


Posets with a Maximum

Let $M_P = \{x \in P \mid \text{if } y \lessdot x \text{ then } y \text{ is minimal}\}$



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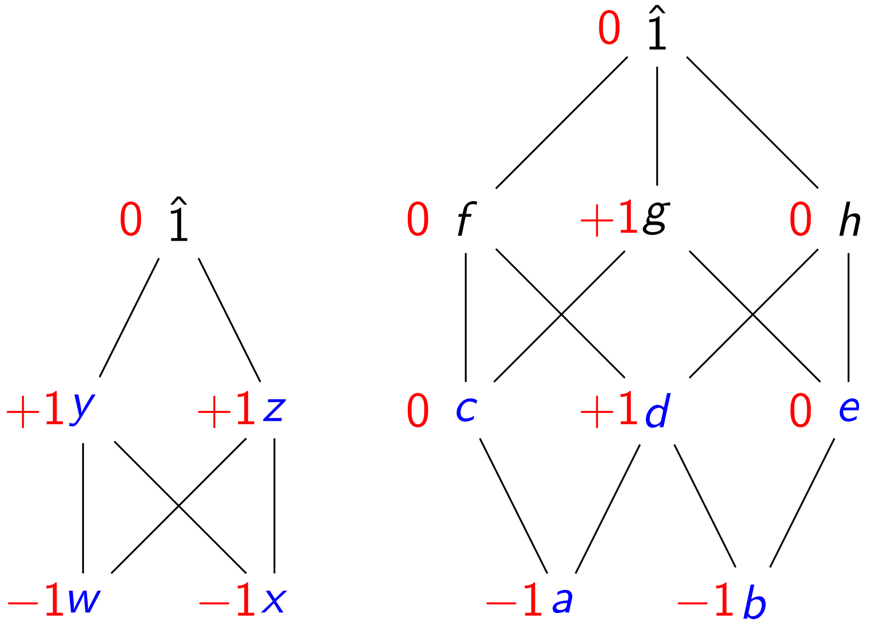
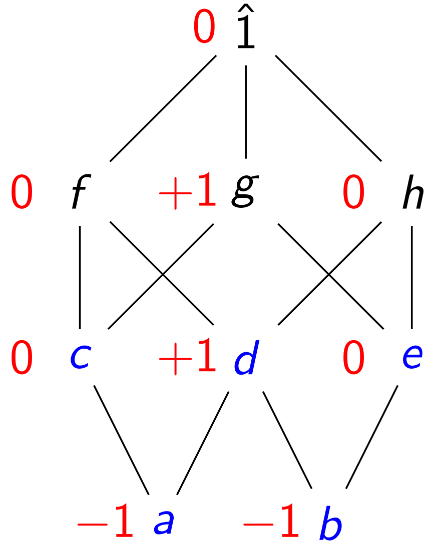


Proposition

If P has a Gorenstein labeling, then M_P is acyclic.

Posets with a Maximum

A connected downset C satisfies the **connected components condition** or **cc condition** if the number of connected components of M_P that have nonempty intersection with C is equal to the number of connected components of $C \cap M_P$.



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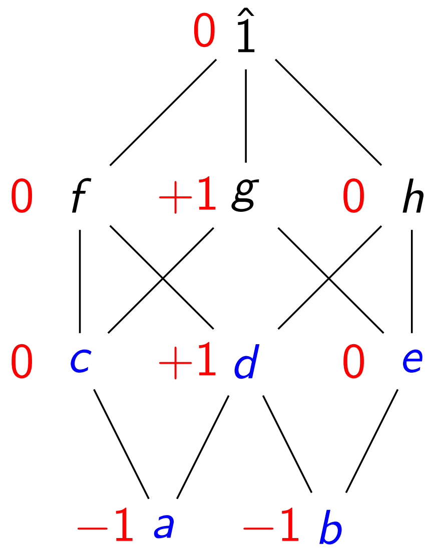

The downset $\{a, b, c, e, g\}$ does not satisfy the cc condition. M_P has only one component, but $M_P \cap \{a, b, c, e, g\} = \{a, c, b, e\}$ which has two components.

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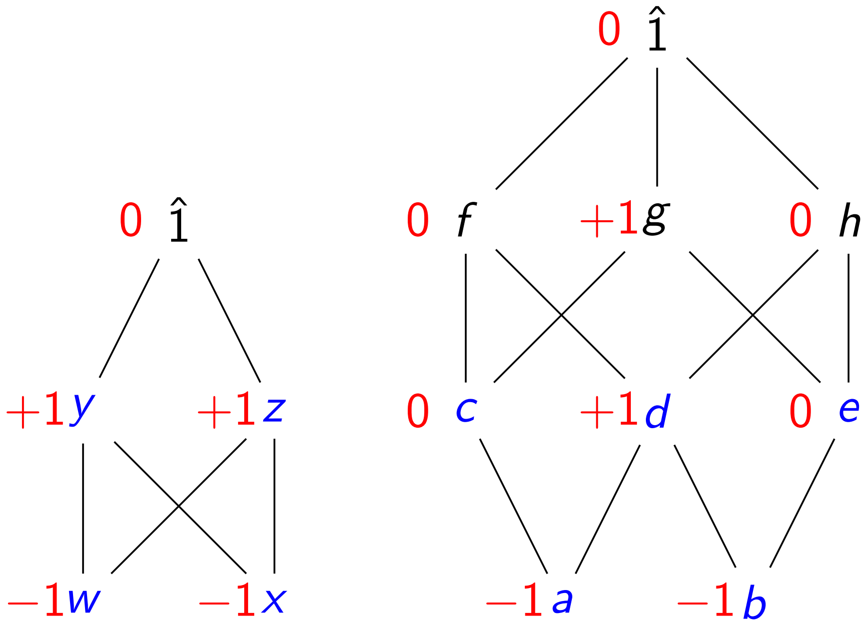
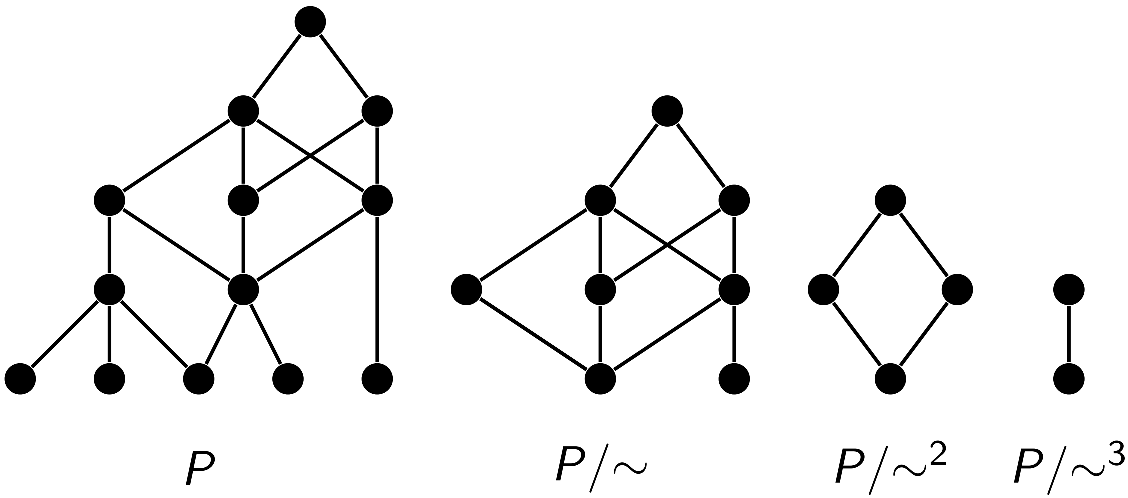


Proposition

If P has a Gorenstein labeling, then all connected downsets of P satisfy the cc condition.

Quotients Posets

Let P be a poset with a $\hat{1}$. We let $P/\sim$ denote **quotient poset** obtained by merging the connected components of M_P.



An Algorithm for Posets with a Maximum

Input: P a poset with a $\hat{1}$.

Output: A Yes/No decision if U_P is Gorenstein.

Method:

1. While $\ell(P) > 1$
   1.1 Check the truth value of the following statement.

       M_P is acyclic.

       If the statement is false, terminate algorithm and return "No".

   1.2 Check the truth value of the following statement.

       *All connected downsets of $P \setminus M_P$ generated by $cc(M_P)$ or less elements satisfy the cc condition.**

       If the statement is false, terminate algorithm and return "No".

   1.3 Set P to be $P/\!\sim$.
2. Return "Yes".

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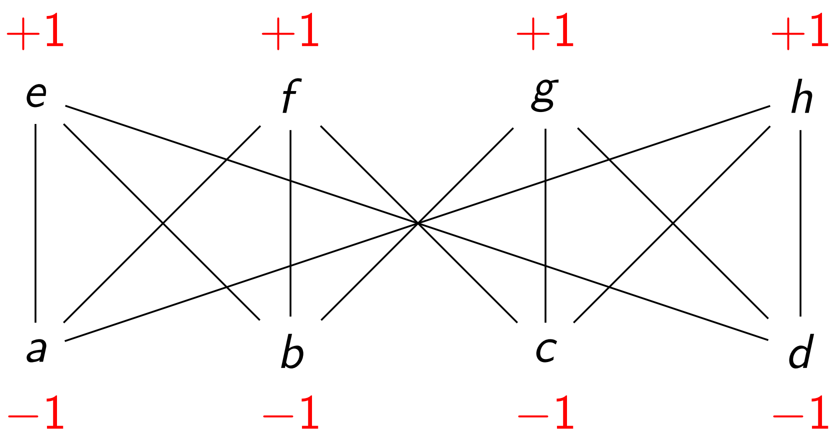
Theorem

The above algorithm is valid and can be made to run in $O(kn^{k+3})$ where $n = |P|$ and k is the number of connected components of M_P.

Posets of Rank 1

It turns out that one can reduce the problem of finding a Gorenstein labeling of a rank 1 poset to looking poset whose Hasse diagram is a biconnected, balanced bipartite graph.



In that case, the minimum elements must have label -1 and the maximum elements must have label 1.

THANK YOU