The Noncrossing Bond Poset of a Graph

C. Matthew Farmer
UNCG

Josh Hallam
Loyola Marymount University

UNCG GANT-C Seminar

March 28, 2019

Introduction

The main idea of the noncrossing bond poset is a combination of two families of posets

- The bond lattice
- The noncrossing partition lattice

We will start by looking at some of the combinatorial properties of these families.

Partially Ordered Sets

A **partially ordered set (poset)** P is set together with a binary relation $\leq$ such that for all $x, y, z \in P$

1. $x \leq x$
2. If $x \leq y$ and $y \leq x$, then $x = y$
3. If $x \leq y$ and $y \leq z$, then $x \leq z$

The minimum element (if it exists) is denoted by $\hat{0}$. The maximum element (if it exists) is denoted by $\hat{1}$.

All the posets we consider will be finite and have a $\hat{0}$, but not necessarily a $\hat{1}$.

Examples of Posets

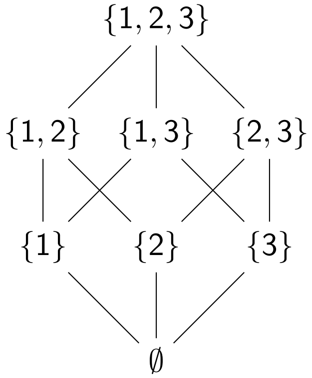
The **chain** C_n with n elements is the set $\{1, 2, \ldots, n\}$ with the normal ordering on $\mathbb{Z}$.

C_3

In C_n, $\hat{0} = 1$ and $\hat{1} = n$.

Examples of Posets

The **boolean algebra** B_n is the set of subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion.

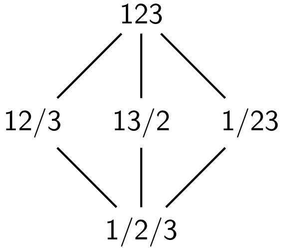


B_3

In B_n, $\hat{0} = \emptyset$ and $\hat{1} = \{1, 2, \ldots, n\}$.

Examples of Posets

The **partition lattice**, Π_n, is the set of partitions of $\{1, 2 \ldots, n\}$ ordered by refinement. That is $B_1/B_2/\cdots/B_k \leq C_1/C_2/\cdots/C_j$ if and only if every block of $C_1/C_2/\cdots/C_j$ is union of blocks in $B_1/B_2/\cdots/B_k$.

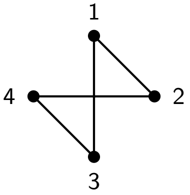


Π_3

Graph Assumptions

All the graphs we will discuss will be finite, simple, and have vertex set $[n] := \{1, 2, \dots, n\}$. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G. We write our edges as ij where $i < j$.

Example Consider the 4-cycle below.

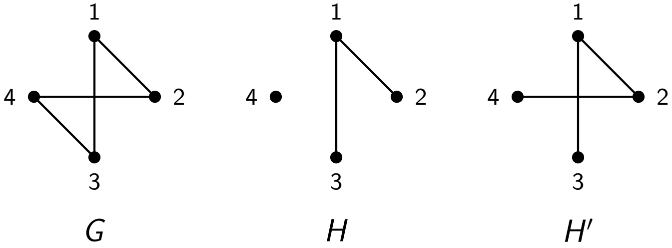


$V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{12, 13, 24, 34\}$.

Bonds

Let G be a graph and let H be a subgraph. We say H is a **spanning subgraph** if H and G have the same vertex set. We say H is **induced** if whenever $u, w \in V(H)$ such that $uw \in E(G)$, then $uw \in E(H)$. We say a spanning subgraph H is a **bond** of G whenever every connected component of H is an induced subgraph of G.

Example

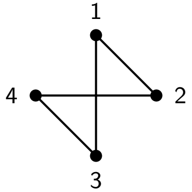


H and H' are both spanning subgraphs of G. H is a bond of G, but H' is not since it is missing the edge 34.

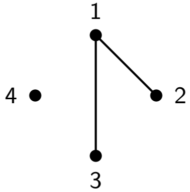
Bonds

Since bonds are spanning subgraphs, they can indentified by the edges they contain.

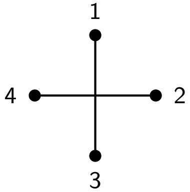
Example



$\{12, 13, 24, 34\}$



$\{12, 13\}$



$\{13, 23\}$

H and H' are both spanning subgraphs of G. H is a bond of G, but H' is not since it is missing the edge 34.

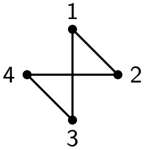
Bond Lattice

Let G be a graph. Ordering the bonds of G by inclusion gives a poset called the **bond lattice** of G. It is denoted by L_G.

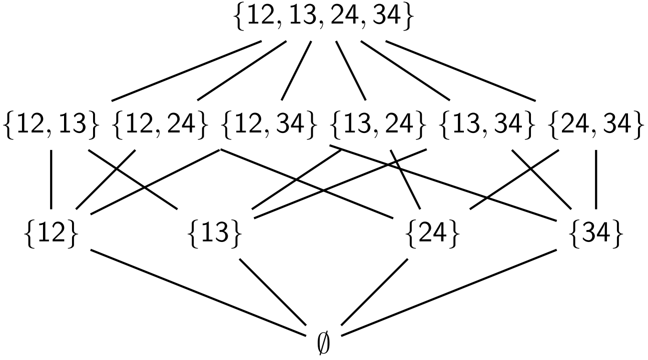
Bond Lattice

Let G be a graph. Ordering the bonds of G by inclusion gives a poset called the **bond lattice** of G. It is denoted by L_G.

Example



G



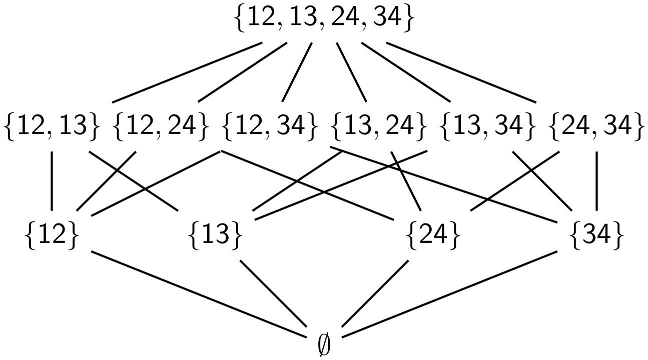
L_G

In L_G, $\hat{0}$ is the bond with no edges and $\hat{1}$ is the graph itself.

Graded Posets

A **chain** in a poset is a totally ordered subset of the poset. We say a chain is **maximal** if is is not contained in any other chain. The **length** of a maximal chain is the number of edges of the chain in the Hasse diagram. If every maximal chain of a poset P has the same length, then we say P is **graded**.

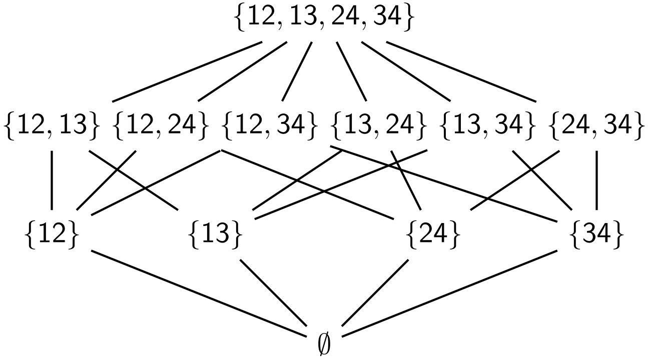
Example



Graded Posets

Given a graded poset, we can define a **rank function** $\rho : P \to \mathbb{Z}$ by setting $\rho(x)$ to be the number of edges need to be traversed from $\hat{0}$ to x in the Hasse diagram of P.

Example



In this example, $\rho(\{12, 13\}) = 2$ and $\rho(\{12, 13, 24, 34\}) = 3$.

The Möbius Function

The **(one–variable) Möbius function** $\mu : P \to \mathbb{Z}$ is defined by

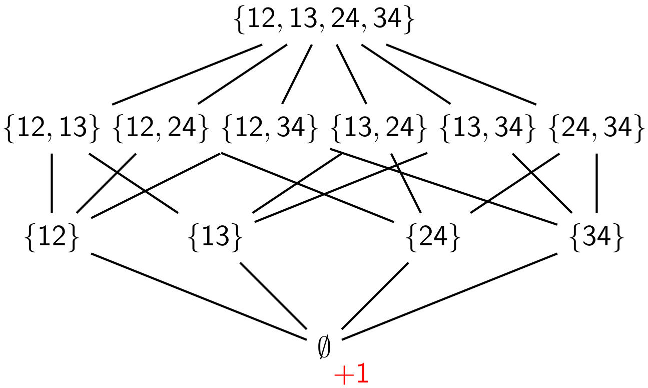
$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{otherwise} \end{cases}$$

The Möbius Function

The **(one–variable) Möbius function** $\mu : P \to \mathbb{Z}$ is defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{otherwise} \end{cases}$$

Example

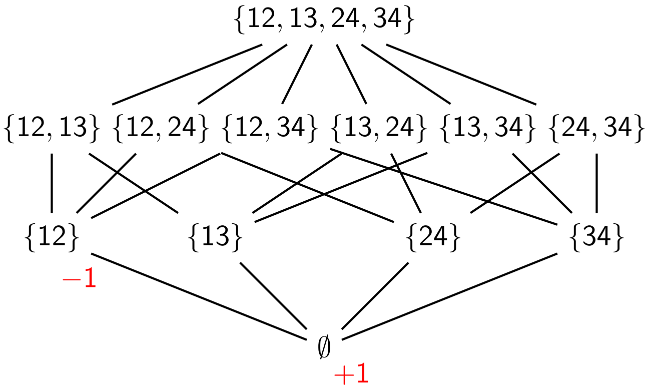


The Möbius Function

The **(one–variable) Möbius function** $\mu : P \to \mathbb{Z}$ is defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{otherwise} \end{cases}$$

Example

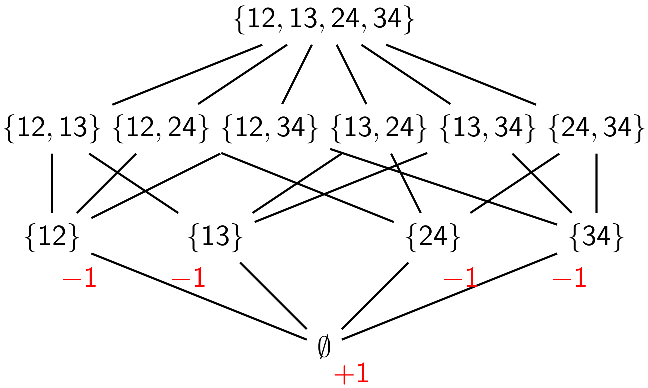


The Möbius Function

The **(one–variable) Möbius function** $\mu : P \to \mathbb{Z}$ is defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{otherwise} \end{cases}$$

Example

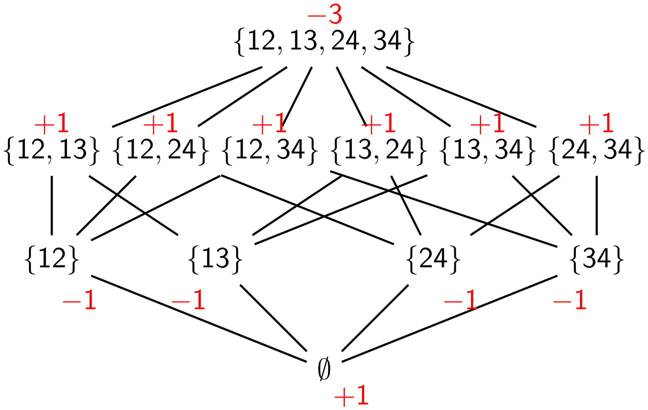


The Möbius Function

The **(one–variable) Möbius function** $\mu : P \to \mathbb{Z}$ is defined by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{otherwise} \end{cases}$$

Example



Characteristic Polynomial

Let P be a graded poset. The **characteristic polynomial** of P is

$$\chi(P, t) = \sum_{x \in P} \mu(x) t^{\rho(P) - \rho(x)}$$

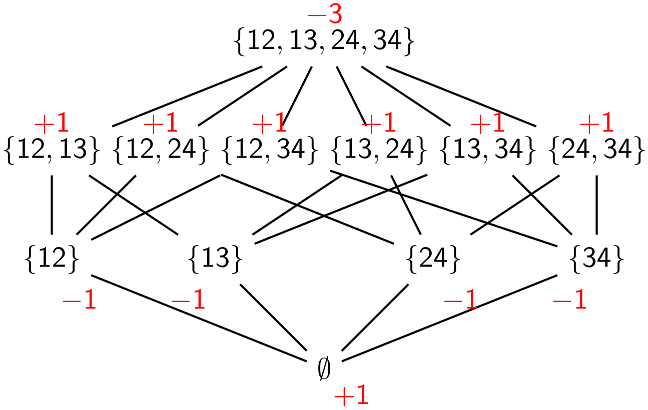
where $\rho(P)$ is the rank of (any) maximal element.

Characteristic Polynomial

Let P be a graded poset. The **characteristic polynomial** of P is

$$\chi(P,t) = \sum_{x \in P} \mu(x) t^{\rho(P)-\rho(x)}$$

where $\rho(P)$ is the rank of (any) maximal element.



So $\chi(P,t) = t^3 - 4t^2 + 6t - 3$

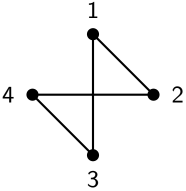
Graph Coloring

Let G be graph. A **proper coloring** G is a function $c : V(G) \to S$ where S is some set of colors such that $uv \in E(G)$ implies $c(u) \neq c(v)$.

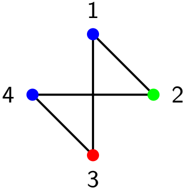
Graph Coloring

Let G be graph. A **proper coloring** G is a function $c : V(G) \to S$ where S is some set of colors such that $uv \in E(G)$ implies $c(u) \neq c(v)$.

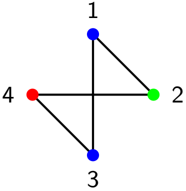
Example



G



Proper



Not Proper

Chromatic Polynomial

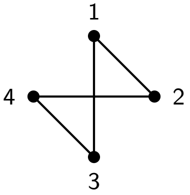
In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G, t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$

Chromatic Polynomial

In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G, t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$



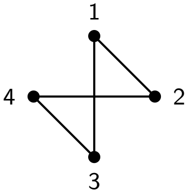
G

So $\chi(G, 0) =$

Chromatic Polynomial

In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G, t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$



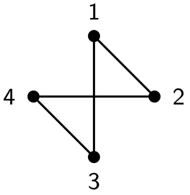
G

So $\chi(G, 0) = 0$,

Chromatic Polynomial

In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G, t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$



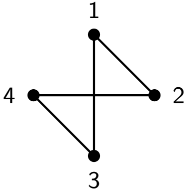
G

So $\chi(G, 0) = 0$, $\chi(G, 1) =$

Chromatic Polynomial

In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G,t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$



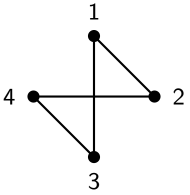
G

So $\chi(G,0) = 0$, $\chi(G,1) = 0$,

Chromatic Polynomial

In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G, t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$



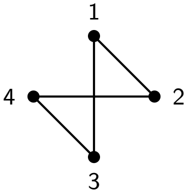
G

So $\chi(G, 0) = 0$, $\chi(G, 1) = 0$, $\chi(G, 2) =$

Chromatic Polynomial

In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G,t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$



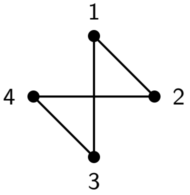
G

So $\chi(G,0) = 0$, $\chi(G,1) = 0$, $\chi(G,2) = 2$,

Chromatic Polynomial

In an attempt to solve the four-color problem, Birkoff introduced a polynomial which keeps track of colorings. Let G be a graph. The **chromatic polynomial** of G is defined by

$$\chi(G,t) = \#(\text{proper colorings of } G \text{ with } t \text{ colors})$$

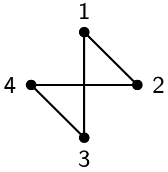


G

So $\chi(G,0) = 0$, $\chi(G,1) = 0$, $\chi(G,2) = 2$, $\chi(G,3) > 2$.

Chromatic Polynomial

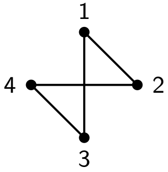
Although it is not obvious, $\chi(G, t)$ is a polynomial in t.



G

Chromatic Polynomial

Although it is not obvious, $\chi(G, t)$ is a polynomial in t.

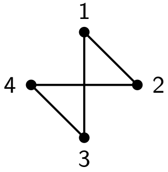


G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color.

Chromatic Polynomial

Although it is not obvious, $\chi(G, t)$ is a polynomial in t.

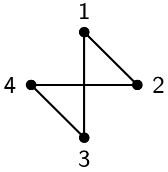


G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are t

Chromatic Polynomial

Although it is not obvious, $\chi(G, t)$ is a polynomial in t.

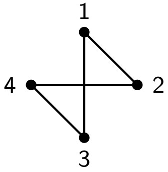


G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)$

Chromatic Polynomial

Although it is not obvious, $\chi(G, t)$ is a polynomial in t.

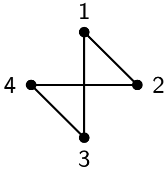


G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)$

Chromatic Polynomial

Although it is not obvious, $\chi(G,t)$ is a polynomial in t.

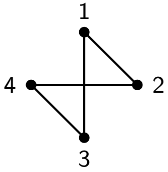


G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)(t-1)$ choices for these colorings.

Chromatic Polynomial

Although it is not obvious, $\chi(G,t)$ is a polynomial in t.



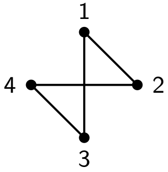
G

We will color the graph by coloring the vertices in the order $1,2,3,4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)(t-1)$ choices for these colorings. Now suppose that that 2 and 3 are different. Then there are

t

Chromatic Polynomial

Although it is not obvious, $\chi(G, t)$ is a polynomial in t.

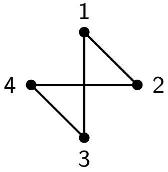


G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)(t-1)$ choices for these colorings. Now suppose that that 2 and 3 are different. Then there are
$t(t-1)$

Chromatic Polynomial

Although it is not obvious, $\chi(G, t)$ is a polynomial in t.

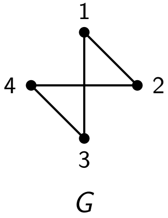


G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)(t-1)$ choices for these colorings. Now suppose that that 2 and 3 are different. Then there are
$t(t-1)(t-2)$

Chromatic Polynomial

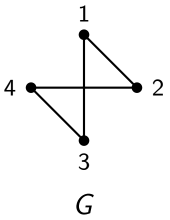
Although it is not obvious, $\chi(G, t)$ is a polynomial in t.



We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)(t-1)$ choices for these colorings. Now suppose that that 2 and 3 are different. Then there are $t(t-1)(t-2)(t-2)$

Chromatic Polynomial

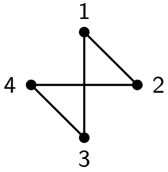
Although it is not obvious, $\chi(G,t)$ is a polynomial in t.



We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)(t-1)$ choices for these colorings. Now suppose that that 2 and 3 are different. Then there are $t(t-1)(t-2)(t-2)$ such colorings.

Chromatic Polynomial

Although it is not obvious, $\chi(G, t)$ is a polynomial in t.



G

We will color the graph by coloring the vertices in the order $1, 2, 3, 4$. Lets consider the case where 2 and 3 are the same color. Then there are $t(t-1)(1)(t-1)$ choices for these colorings. Now suppose that that 2 and 3 are different. Then there are $t(t-1)(t-2)(t-2)$ such colorings.

$$
\begin{aligned}
\chi(G, t) &= t(t-1)(t-1) + t(t-1)(t-2)^2 \\
&= t(t^3 - 4t^2 + 6t - 3) \\
&= t\chi(L_G, t)
\end{aligned}
$$

Chromatic Polynomial

Theorem

Let G be graph with k connected components. Then

$$\chi(G, t) = t^k \chi(L_G, t).$$

Chromatic Polynomial

Theorem

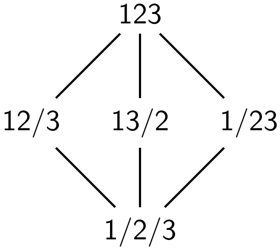
Let G be graph with k connected components. Then

$$\chi(G, t) = t^k \chi(L_G, t).$$

Remark Stanley introduced a symmetric function analogue of the chromatic polynomial called the **chromatic symmetric function**. It can also be read off from the bond lattice of the graph.

The Partition Lattice

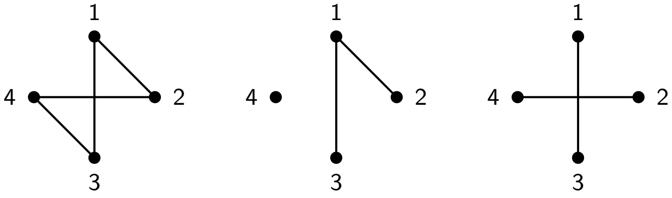
The **partition lattice**, Π_n, is the set of partitions of $\{1, 2 \ldots, n\}$ ordered by refinement. That is $B_1/B_2/\cdots/B_k \leq C_1/C_2/\cdots/C_j$ if and only if every block of $C_1/C_2/\cdots/C_j$ is union of blocks in $B_1/B_2/\cdots/B_k$.



Π_3

Set Partitions and Bonds

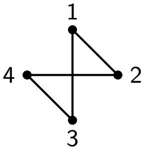
Let G be a graph. For each bond H we can associate a set partition by letting i, j be in the same block of the partition if and only if i and j are in the same connected component of H.



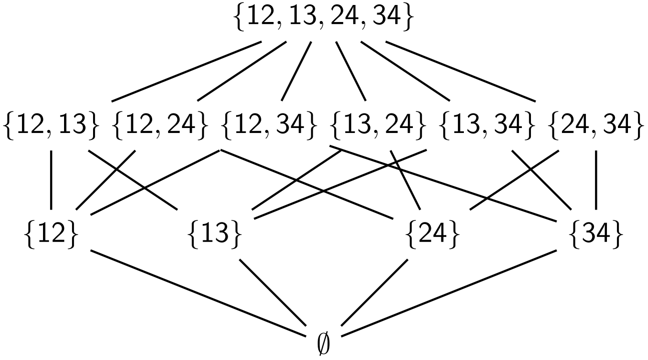
1234 $\qquad$ $123/4$ $\qquad$ $13/24$

Set Partitions and Bonds

Example



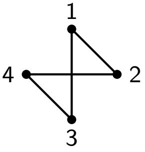
G



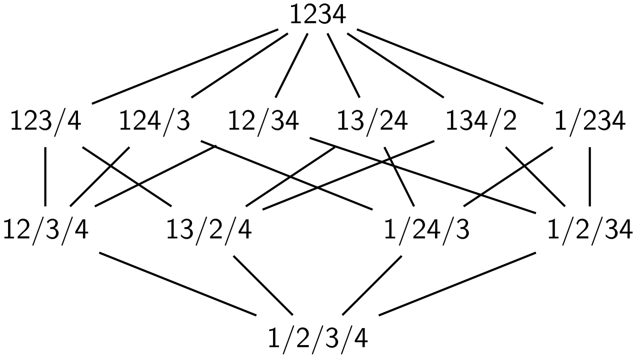
L_G

Set Partitions and Bonds

Example



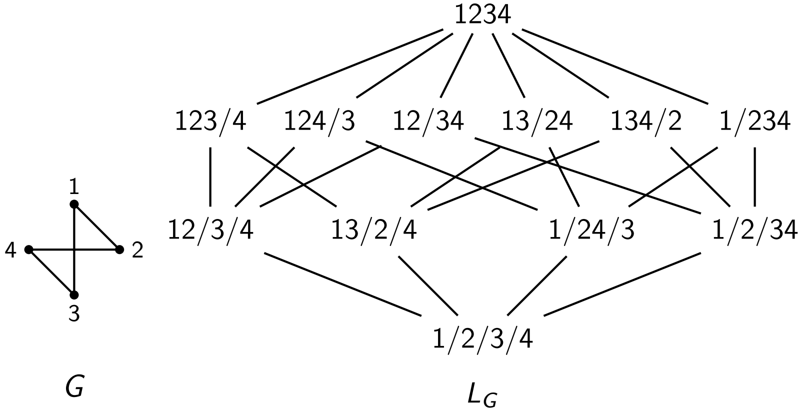
G



L_G

Set Partitions and Bonds

Example



So you can see that L_G is a subposet of Π_n where n is the number of vertices of G. In fact, Π_n is (isomorphic to) the bond lattice of the complete graph.

The Noncrossing Partition Lattice

Let $\pi = B_1/B_2/\cdots/B_k$ be a set partition. We say π is **crossing** if there exists blocks $a, c \in B_i$ and $b, d \in B_j$ with $a < b < c < d$ otherwise we say it is **noncrossing**. For example, $124/35/6$ is crossing whereas $125/34/67$ is noncrossing.

The Noncrossing Partition Lattice

Let $\pi = B_1/B_2/\cdots/B_k$ be a set partition. We say π is **crossing** if there exists blocks $a, c \in B_i$ and $b, d \in B_j$ with $a < b < c < d$ otherwise we say it is **noncrossing**. For example, 124/35/6 is crossing whereas 125/34/67 is noncrossing. The **noncrossing partition lattice**, $\mathcal{NC}_n$ is the poset obtained by removing all the crossing partitions from the partition lattice.

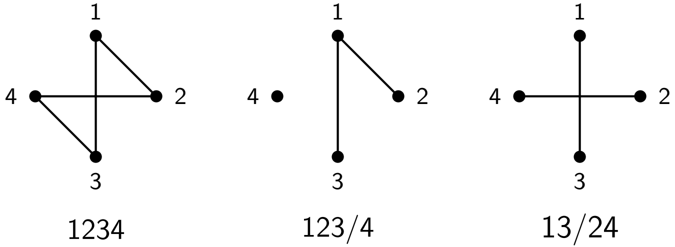
The Noncrossing Partition Lattice

The noncrossing partition lattice has many nice combinatorial properties. For $\mathcal{NC}_{n+1}$,

- It has a Catalan number of objects.
- The Möbius value of the maximal element is a Catalan number. More generally, the Möbius function of any element is a product of Catalan numbers.
- It is rank-symmetric. That is, the number of elements of rank i is the same as the number of elements of rank $n-i$
- The number of maximal chains is $(n+1)^{n-1}$ (which is the number of trees on $n+1$ labeled vertices).

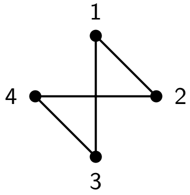
Noncrossing Bonds

Let H be a bond of a graph G. We say H is a **noncrossing bond** if its corresponding partition is noncrossing.



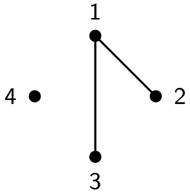
Noncrossing Bonds

Let H be a bond of a graph G. We say H is a **noncrossing bond** if its corresponding partition is noncrossing.

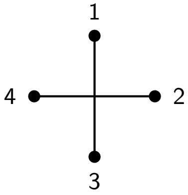


1234

noncrossing



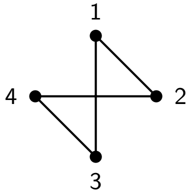
123/4



13/24

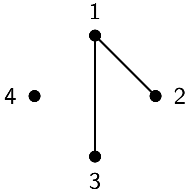
Noncrossing Bonds

Let H be a bond of a graph G. We say H is a **noncrossing bond** if its corresponding partition is noncrossing.



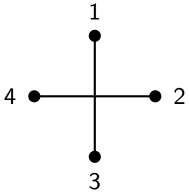
1234

noncrossing



123/4

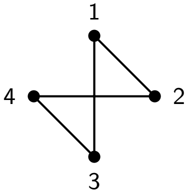
noncrossing



13/24

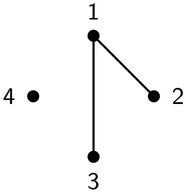
Noncrossing Bonds

Let H be a bond of a graph G. We say H is a **noncrossing bond** if its corresponding partition is noncrossing.



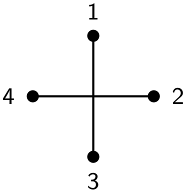
1234

noncrossing



123/4

noncrossing

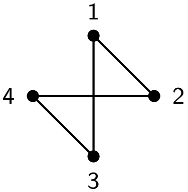


13/24

crossing

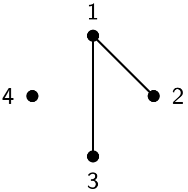
Noncrossing Bonds

Let H be a bond of a graph G. We say H is a **noncrossing bond** if its corresponding partition is noncrossing.



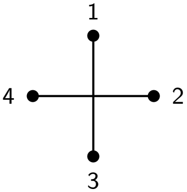
1234

noncrossing



123/4

noncrossing



13/24

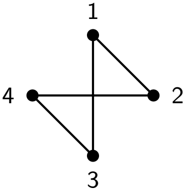
crossing

Notice that crossing bond is not the same as having edges that cross!

Noncrossing Bonds

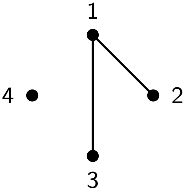
Lemma

A bond is crossing if and only if there are edges in different connected components which cross.



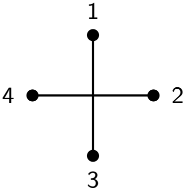
1234

noncrossing



123/4

noncrossing



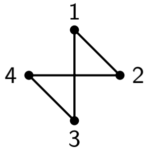
13/24

crossing

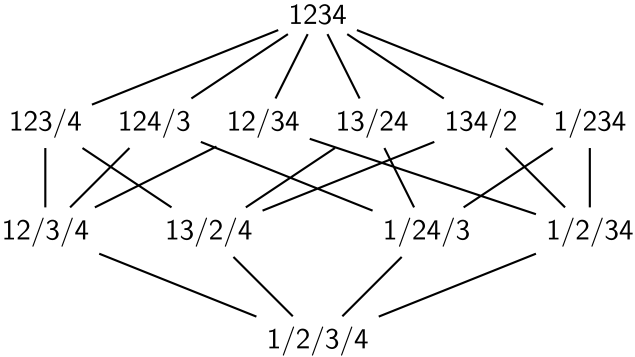
Noncrossing Bond Poset

Let G be a graph. The **noncrossing bond poset** of G, denoted by $\mathcal{NC}_G$ is the poset obtained by removing any crossing bonds from L_G.

Example



G

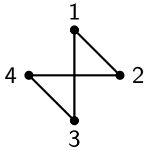


L_G

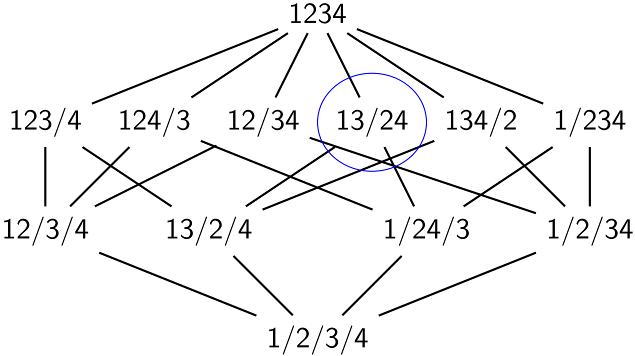
Noncrossing Bond Poset

Let G be a graph. The **noncrossing bond poset** of G, denoted by $\mathcal{NC}_G$ is the poset obtained by removing any crossing bonds from L_G.

Example



G

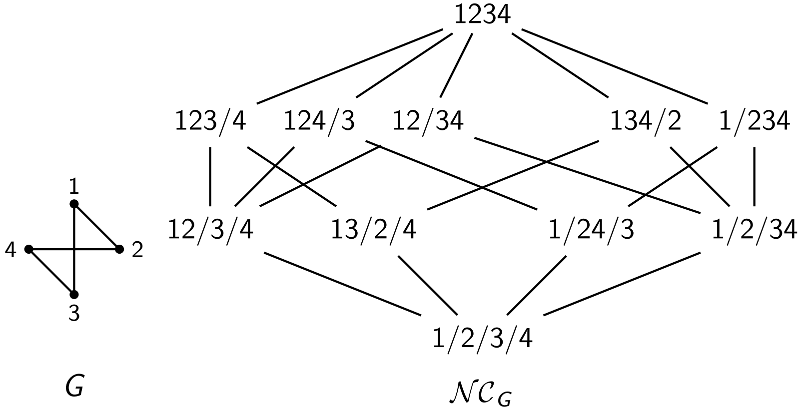


L_G

Noncrossing Bond Poset

Let G be a graph. The **noncrossing bond poset** of G, denoted by $\mathcal{NC}_G$ is the poset obtained by removing any crossing bonds from L_G.

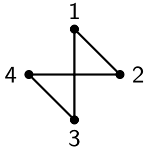
Example



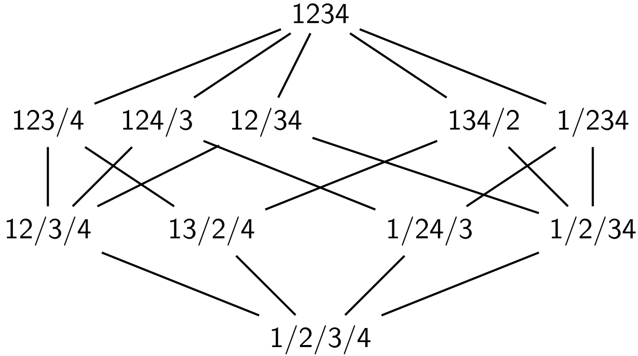
Noncrossing Bond Poset

Let G be a graph. The **noncrossing bond poset** of G, denoted by $\mathcal{NC}_G$ is the poset obtained by removing any crossing bonds from L_G.

Example



G



$\mathcal{NC}_G$

Remark: The noncrossing partition lattice is the noncrossing bond poset of the complete graph.

Properties of the Noncrossing Bond Poset

We can think of $\mathcal{NC}_G$ is a combination of the bond lattice and the noncrossing partition lattice. So it seems natural to ask what properties of these two posets carry over to $\mathcal{NC}_G$.

1. Is $\mathcal{NC}_G$ always a lattice?

Properties of the Noncrossing Bond Poset

We can think of $\mathcal{NC}_G$ is a combination of the bond lattice and the noncrossing partition lattice. So it seems natural to ask what properties of these two posets carry over to $\mathcal{NC}_G$.

1. Is $\mathcal{NC}_G$ always a lattice? no

Properties of the Noncrossing Bond Poset

We can think of $\mathcal{NC}_G$ is a combination of the bond lattice and the noncrossing partition lattice. So it seems natural to ask what properties of these two posets carry over to $\mathcal{NC}_G$.

1. Is $\mathcal{NC}_G$ always a lattice? no
2. Is $\mathcal{NC}_G$ always graded?

Properties of the Noncrossing Bond Poset

We can think of $\mathcal{NC}_G$ is a combination of the bond lattice and the noncrossing partition lattice. So it seems natural to ask what properties of these two posets carry over to $\mathcal{NC}_G$.

1. Is $\mathcal{NC}_G$ always a lattice? no
2. Is $\mathcal{NC}_G$ always graded? no

Properties of the Noncrossing Bond Poset

We can think of $\mathcal{NC}_G$ is a combination of the bond lattice and the noncrossing partition lattice. So it seems natural to ask what properties of these two posets carry over to $\mathcal{NC}_G$.

1. Is $\mathcal{NC}_G$ always a lattice? no
2. Is $\mathcal{NC}_G$ always graded? no
3. How does the Möbius function and characteristic polynomial behave in $\mathcal{NC}_G$? for some graphs, in a way similar to L_G

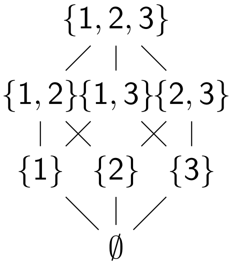
Properties of the Noncrossing Bond Poset

We can think of $\mathcal{NC}_G$ is a combination of the bond lattice and the noncrossing partition lattice. So it seems natural to ask what properties of these two posets carry over to $\mathcal{NC}_G$.

1. Is $\mathcal{NC}_G$ always a lattice? no
2. Is $\mathcal{NC}_G$ always graded? no
3. How does the Möbius function and characteristic polynomial behave in $\mathcal{NC}_G$? for some graphs, in a way similar to L_G

The Join and Meet

Let P be a poset and let $x, y \in P$. The **meet** of x and y, if it exists, is an element z such that $z \leq x, y$ and if $w \leq x, y$ then $w \leq z$. The meet is denoted by $x \wedge y$. The **join** of x and y, if it exists, is an element z such that $x, y \leq z$ and if $x, y \leq w$ then $z \leq w$. The join is denoted by $x \vee y$.



B_3

In B_n, $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.

Lattices

A poset is called a **lattice** if every pair of elements has a meet and a join. Examples include

Lattices

A poset is called a **lattice** if every pair of elements has a meet and a join. Examples include

- The Chain
- Boolean Algebra
- Bond Lattice.

Lattices

A poset is called a **lattice** if every pair of elements has a meet and a join. Examples include

- The Chain
- Boolean Algebra
- Bond Lattice.
  - The intersection of any two bonds is a bond so the meet is the intersection of the bonds.

Lattices

A poset is called a **lattice** if every pair of elements has a meet and a join. Examples include

- The Chain
- Boolean Algebra
- Bond Lattice.
    - The intersection of any two bonds is a bond so the meet is the intersection of the bonds.
    - The join is **not** the union since the union of two bonds need not be a bond.

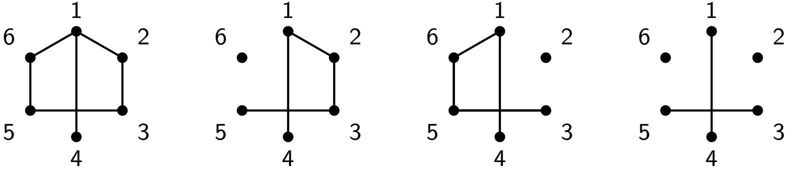
Lattices

A poset is called a **lattice** if every pair of elements has a meet and a join. Examples include

- The Chain
- Boolean Algebra
- Bond Lattice.
  - The intersection of any two bonds is a bond so the meet is the intersection of the bonds.
  - The join is **not** the union since the union of two bonds need not be a bond.
- The Partition Lattice
- The Noncrossing Partition Lattice

Is the noncrossing bond poset a lattice?

In general, the answer is no. The main issue is that the intersection of noncrossing bonds does not need to be noncrossing. For example,

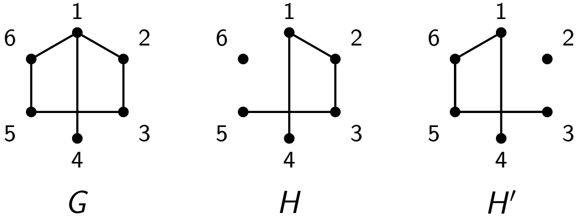


G $\quad$ H $\quad$ H' $\quad$ $H \cap H'$

Here the only noncrossing bonds below H and H' are $\{14\}$, $\{35\}$, and $\emptyset$.

Crossing Closed

Let G be a graph. We say G is **crossing closed** if whenever ac, bd are crossing edges in G, there is a unique (with respect to inclusion) induced connected component containing both ac and bd.



Here G is not crossing closed.

Crossing Closed

Theorem (Farmer-H)

Let G be a connected graph. Then $\mathcal{NC}_G$ is a lattice if and only if G is crossing closed. Moreover, if G is crossing closed, $H \wedge H' = H \cap H'$.

Crossing Closed

Theorem (Farmer-H)

Let G be a connected graph. Then $\mathcal{NC}_G$ is a lattice if and only if G is crossing closed. Moreover, if G is crossing closed, $H \wedge H' = H \cap H'$.

Examples of graphs that are crossing closed.

- Trees
- Complete graphs
- Graphs with no crossings.

Crossing Closed

Theorem (Farmer-H)

Let G be a connected graph. Then $\mathcal{NC}_G$ is a lattice if and only if G is crossing closed. Moreover, if G is crossing closed, $H \wedge H' = H \cap H'$.

Examples of graphs that are crossing closed.

- Trees
- Complete graphs
- Graphs with no crossings.

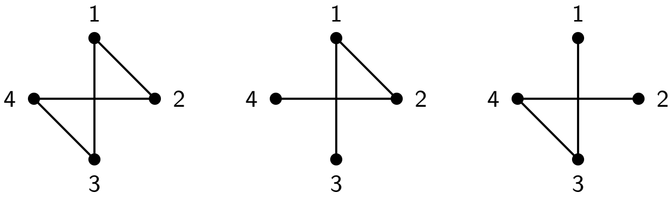
The Möbius Function of $\mathcal{NC}_G$

Earlier we saw that the characteristic polynomial of L_G was (up to a power of t) the chromatic polynomial of G. There is a combinatorial formula fo the coefficients of the chromatic polynomial due to Whitney.

The Möbius Function of $\mathcal{NC}_G$

Earlier we saw that the characteristic polynomial of L_G was (up to a power of t) the chromatic polynomial of G. There is a combinatorial formula fo the coefficients of the chromatic polynomial due to Whitney.

Let G be a graph and let $\trianglelefteq$ be a total order on the edges. A **broken circuit** of G is a subgraph of G obtained by removing the smallest edge of a cycle of G.



G H H'

With the odering $12 < 13 < 24 < 34$, H' is a broken circuit and H is not.

The Möbius Function of $\mathcal{NC}_G$

A **non-broken circuit (NBC) set** of a graph G is a spanning subgraph which does not contain any broken circuits.

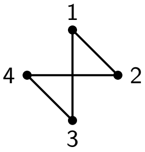
Theorem (Whitney 1932)

Let G be a graph. Then

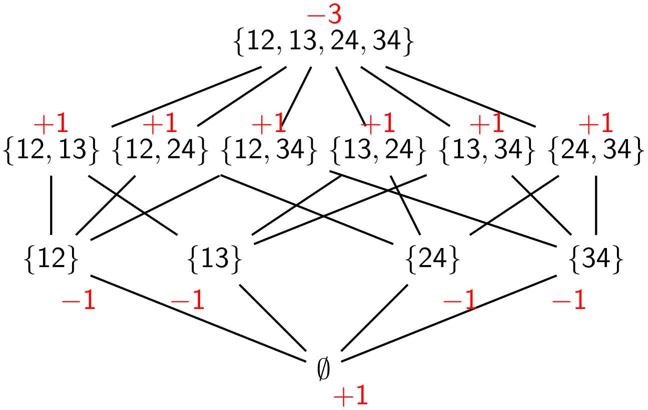
$$\chi(L_G, t) = \sum_{k \geq 0} (-1)^k nbc_k(G) t^{\rho(L)-k}$$

where $nbc_k(G)$ is the number of NBC sets of G with k edges.

The Möbius Function of $\mathcal{NC}_G$



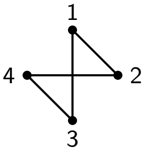
G



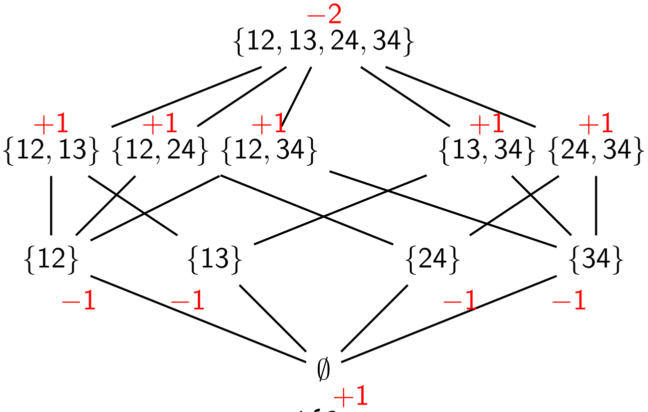
L_G

So $\chi(L_G, t) = t^3 - 4t^2 + 6t - 3$

The Möbius Function of $\mathcal{NC}_G$



G

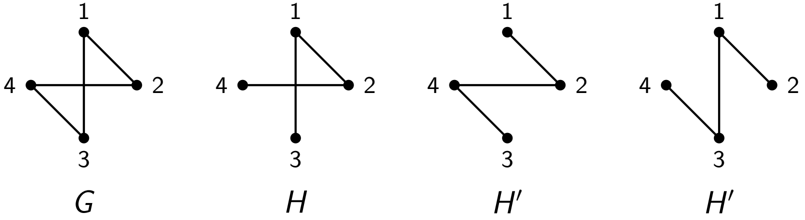


$\mathcal{NC}_G$

So $\chi(\mathcal{NC}_G, t) = t^3 - 4t^2 + 5t - 2$

The Möbius Function of $\mathcal{NC}_G$

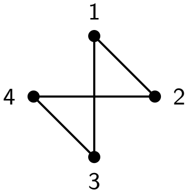
We say an NBC set is a **noncrossing NBC set** if none of its edges cross.



With the odering $12 < 13 < 24 < 34$, H, H', and H'' are NBC sets. However, only H' and H'' are noncrossing.

Upper Crossing Closed

Let G be a graph which is crossing closed and let $\trianglelefteq$ be a total order on the edges of G. We say G is **upper crossing** with respect to $\trianglelefteq$ if whenever e and f are edges which cross, the unique minimal induced connected component containing e and f contains an edge smaller than both e and f.



G is upper crossing closed with respect to ordering $12 < 13 < 24 < 34$, G, but not with respect to $13 < 12 < 24 < 34$.

The Möbius Function of $\mathcal{NC}_G$

Theorem (Farmer-H)

Let G be a graph which is upper crossing closed with repsect to $\trianglelefteq$. Then for $H \in \mathcal{NC}_G$,

$$\mu(H) = (-1)^{\rho(H)} \#(\textit{noncrossing NBC sets which span } H)$$

The Möbius Function of $\mathcal{NC}_G$

Theorem (Farmer-H)

Let G be a graph which is upper crossing closed with repsect to $\trianglelefteq$. Then for $H \in \mathcal{NC}_G$,

$$\mu(H) = (-1)^{\rho(H)} \#(\text{noncrossing NBC sets which span } H)$$

Moreover, if $\mathcal{NC}_G$ is connected, graded and G has n vertices, then

$$\chi(\mathcal{NC}_G, t) = \sum_{k \geq 0} (-1)^k ncnbc_k(G) t^{n-k}$$

where $ncnbc_k(G)$ is the number of noncrossing NBC sets of G with k edges.

Other Properties of $\mathcal{NC}_G$

In addition to what we have discussed, we also have the following.

1. Every poset has an associated simplicial complex called the **order complex**. The (reduced) Euler characteristic of this complex is the Möbius value of the top element of the poset. For some posets, the order complex is **shellable**, which is a nice combinatorial proprerty that implies the order complex is (homotopic to) a wedge of spheres. We have necessary conditions (but not sufficient) to show for certain G, the order complex of $\mathcal{NC}_G$ is shellable.

Other Properties of $\mathcal{NC}_G$

In addition to what we have discussed, we also have the following.

1. Every poset has an associated simplicial complex called the **order complex**. The (reduced) Euler characteristic of this complex is the Möbius value of the top element of the poset. For some posets, the order complex is **shellable**, which is a nice combinatorial proprerty that implies the order complex is (homotopic to) a wedge of spheres. We have necessary conditions (but not sufficient) to show for certain G, the order complex of $\mathcal{NC}_G$ is shellable.
2. The dihedral group acts on the graph in way which preserves the structure of $\mathcal{NC}_G$.

Other Properties of $\mathcal{NC}_G$

In addition to what we have discussed, we also have the following.

1. Every poset has an associated simplicial complex called the **order complex**. The (reduced) Euler characteristic of this complex is the Möbius value of the top element of the poset. For some posets, the order complex is **shellable**, which is a nice combinatorial proprerty that implies the order complex is (homotopic to) a wedge of spheres. We have necessary conditions (but not sufficient) to show for certain G, the order complex of $\mathcal{NC}_G$ is shellable.
2. The dihedral group acts on the graph in way which preserves the structure of $\mathcal{NC}_G$.

Open Questions

1. When is $\mathcal{NC}_G$ graded?

Open Questions

1. When is $\mathcal{NC}_G$ graded?
2. Which graphs are such that μ counts the noncrossing NBC sets?

Open Questions

1. When is $\mathcal{NC}_G$ graded?
2. Which graphs are such that μ counts the noncrossing NBC sets?
   - In addition to upper crossing closed graphs, graphs with exactly one crossing also have this property.

Open Questions

1. When is $\mathcal{NC}_G$ graded?
2. Which graphs are such that μ counts the noncrossing NBC sets?
   - In addition to upper crossing closed graphs, graphs with exactly one crossing also have this property.
   - There are graphs where μ cannot count the noncrossing NBC sets with any ordering of the edges.

Open Questions

1. When is $\mathcal{NC}_G$ graded?
2. Which graphs are such that μ counts the noncrossing NBC sets?
   - In addition to upper crossing closed graphs, graphs with exactly one crossing also have this property.
   - There are graphs where μ cannot count the noncrossing NBC sets with any ordering of the edges.
3. When is the order complex of $\mathcal{NC}_G$ shellable?

THANK YOU!