The increasing chain complex of a poset and the top-heavy conjecture

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The **increasing chain complex** will serve as a combinatorial model for these numbers.

The Top-Heavy Conjecture (for vector spaces)

Top-Heavy Conjecture (for vector spaces): Let V be a vector space of dimension n and let E be a finite set of vectors which span V. Let W_i be the number of subspaces of dimension i which are spanned by a subset of E. Then

$$W_i \leq W_{n-i} \text{ for all } i \leq \frac{n}{2}.$$

The Top-Heavy Conjecture Examples

Example: Let $E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and let V the span of E.

Then $\dim V = 3$ and

$$W_0 = 1, W_1 = 3, W_2 = 3, W_3 = 1.$$

Note that

$$W_0 \leq W_3 \text{ and } W_1 \leq W_2$$

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More generally, if E is a set of linearly independent vectors which spans a vector space of dimension n, then

$$W_i = \binom{n}{i} = \binom{n}{n-i} = W_{n-i}.$$

So, $W_i \leq W_{n-i}$.

The Top-Heavy Conjecture Examples

Example: Let $E = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

Let V be the subspace that E spans. Then $\dim V = 3$ and

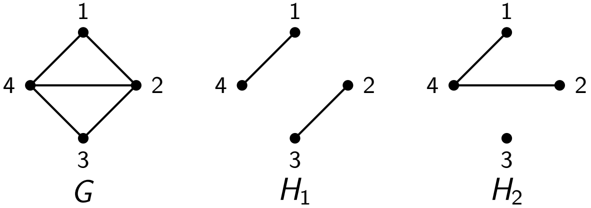
$$W_0 = 1, W_1 = 5, W_2 = 6, W_3 = 1.$$

Note that

$$W_0 \leq W_3 \text{ and } W_1 \leq W_2$$

Flats in Graphs

Let G be a finite graph. A **spanning subgraph** of G is subgraph H such that $V(H) = V(G)$. A **flat** of G is a spanning subgraph H with the property that if $e \in E(G) \setminus E(H)$, then adding e to H decreases the number of connected components of H.



H_1 is a flat of G, but H_2 is not since we can add 12 without decreasing number of connected components.

Flats in Graphs

Given a flat, H, its **rank** is the maximal number of edges is a forest of H. Let W_i be the number of flats of G of rank i.

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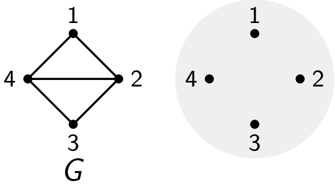

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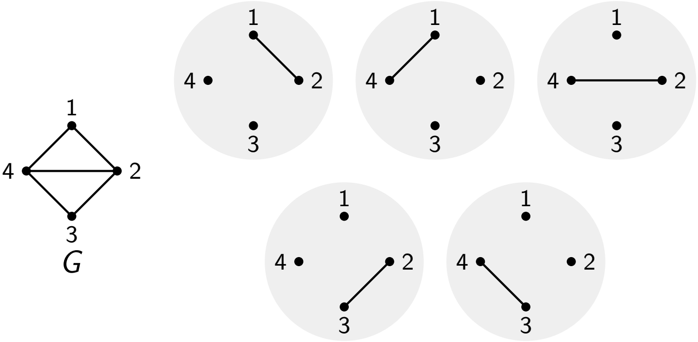

$$W_0 = 1, W_1 = 5$$

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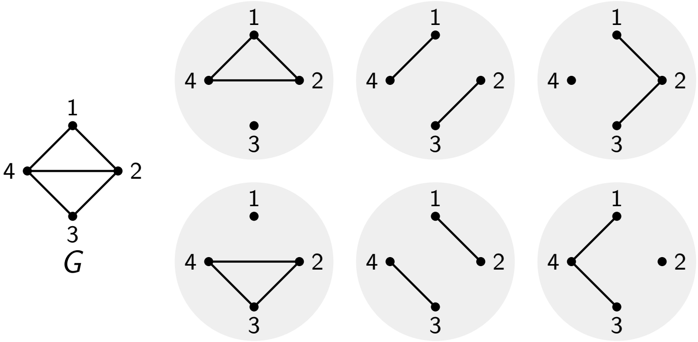

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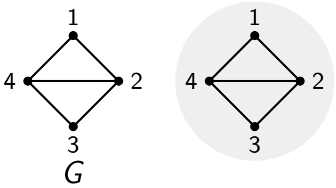

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Top-Heavy Conjecture (for graphs)

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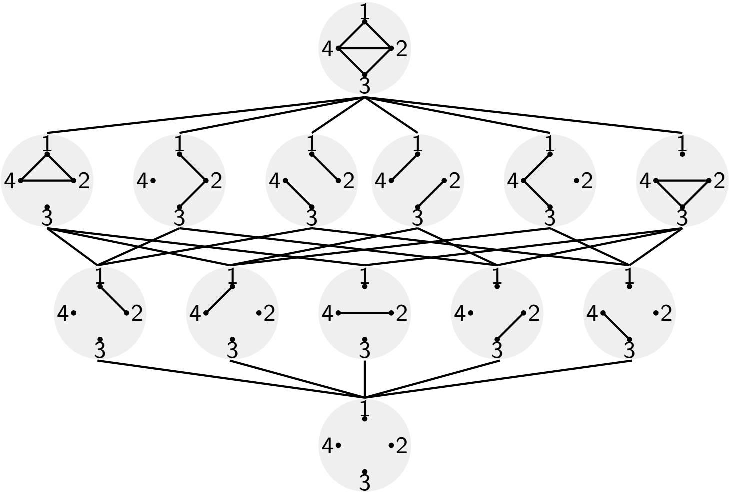
This conjecture is a special case of the conjecture we mentioned about vector spaces which is in turn a special case of the general Top-Heavy Conjecture for matroids. The vector space version (and hence the graph version) was resolved by June Huh and Botong Wang in 2017, but the general case remains open.

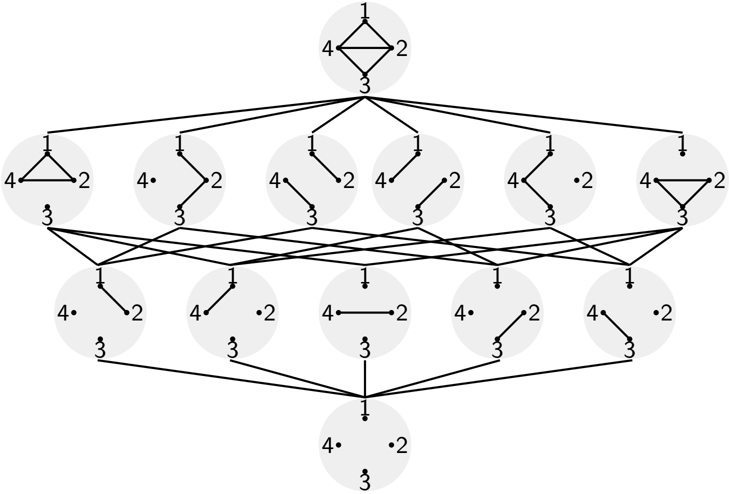
Lattice of Flats

Ordering the flats of graphs by (edge set) inclusion gives a poset which turns out to be a lattice.



Graded Posets

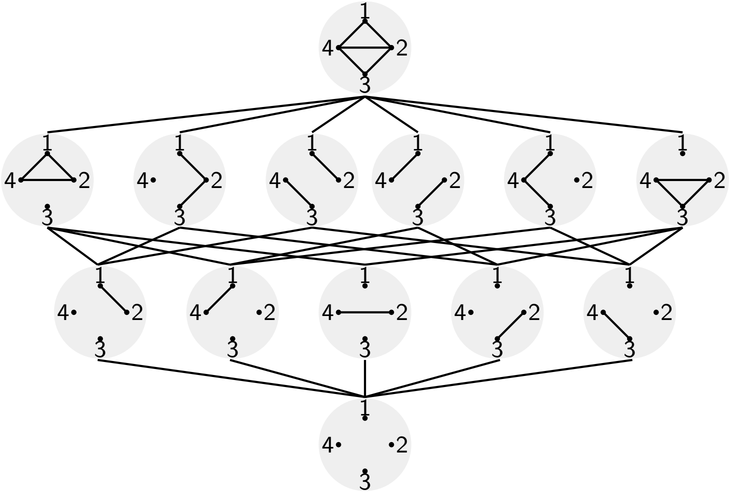
Let P be a poset with a minimum element $\hat{0}$ and a maximal element $\hat{1}$. We say P is **graded** if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length.



Rank Function

If P is graded, then it has a **rank function** defined by

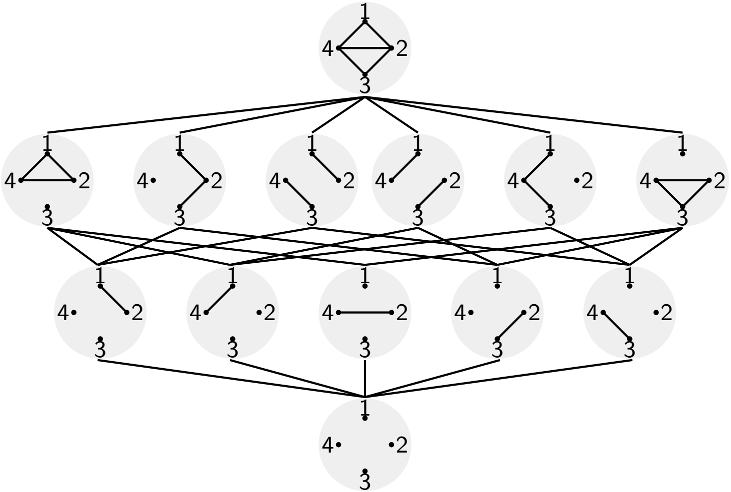
$$\rho(x) = \text{the length of a maximal chain from } \hat{0} \text{ to } x.$$



$\rho(H) =$ maximum number of edges in a forest of H.

Whitney Numbers of the Second Kind

Given a ranked poset, P, the i^{th} **Whitney number of the second kind**, W_i, is the number of elements of rank i.



$$W_0 = 1, W_1 = 5, W_2 = 6, W_4 = 1$$

The Top-Heavy Conjecture (for lattice of flats)

Let L be the lattice of flats of a matroid and let W_i be the i^{th} Whitney number of the second kind. Then

$$W_i \leq W_{n-i} \text{ for } i \leq \frac{n}{2}$$

where n is the rank of L.

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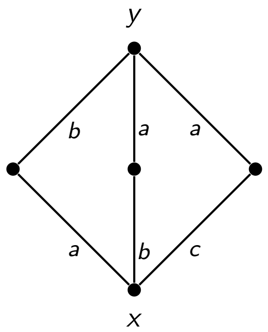
Note that one advantage of this formulation is that one can ask the same question for any graded poset.

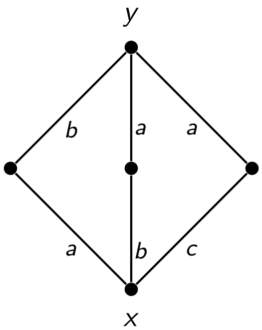
Edge Labelings

Let P be a poset. An **edge labeling** of P is a labeling of the Hasse diagram of P from a set of labels, which are totally ordered. Let $x, y \in P$. A maximal chain from x to y is called **increasing** if the labels along the chain from x to y are strictly increasing.



Edge Labelings

An edge labeling is an **ER-labeling** if for all $x \leq y$, there exists a unique maximal chain from x to y which is increasing.



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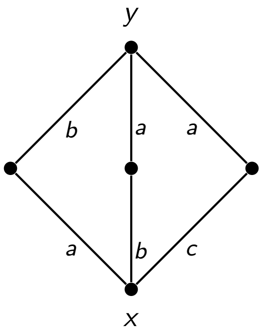

ER-labeling where originally introduced by Stanley to compute the Möbius function of posets. It was later shown by Björner that, by requiring a little bit more, these labelings could be used to understand the topology of the order complex of a poset.

Increasing Chains and Whitney Numbers

Observation

Let P be a poset with an ER-labeling. Then for all $x \in P$, there is a unique increasing maximal chain from $\hat{0}$ to x.

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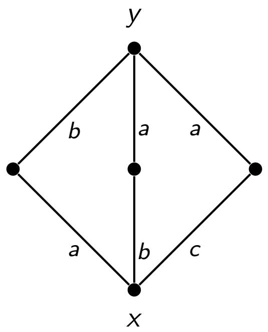
Thus the Whitney numbers of the second kind count increasing chains.

For many labelings, the labels along the unique increasing chain from $\hat{0}$ to x completely determine x. In these cases, the Whitney numbers of the second kind count these sequences.

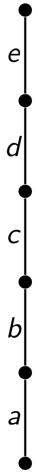
The Rank Two Switching Property

Let P be a poset with an ER-labeling. We say the ER-labeling has the rank two switching property if whenever $x \leq y$ with $\rho(y) - \rho(x) = 2$, if the unique increasing maximal chain is labeled by ab, there is a unique maximal chain labeled by ba.



Increasing Chains and the Rank Two Switching Property

Consider the increasing chain below labeled by *abcde*. Using the rank two switching property, we can find an increasing chain from the bottom labeled by *ace*.



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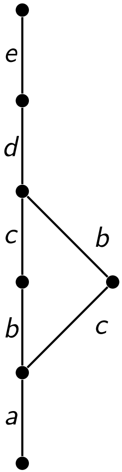

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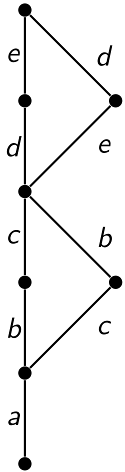

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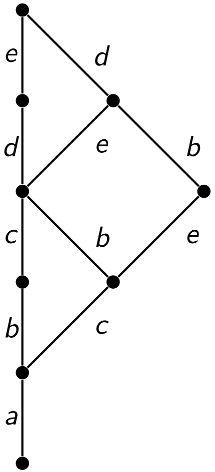

Increasing Chains and the Rank Two Switching Property

Proposition

Let P be a poset with an ER-labeling with the rank-two switching property. If $\{a_1, a_2, \ldots, a_k\}$ is the set of labels along some increasing maximal chain from $\hat{0}$ to some element x, then for any subset of $\{a_1, a_2, \ldots, a_k\}$ there is an increasing maximal chain from $\hat{0}$ to a an element $y \in P$ labeled by this subset of labels.

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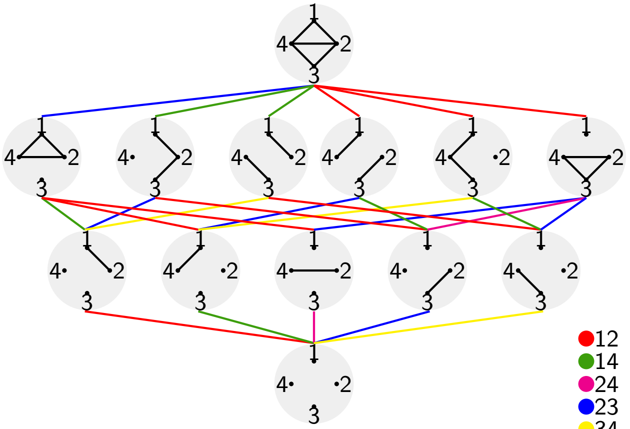
As a result if one considers the collection of subsets of labels that appear along an increasing chains starting at $\hat{0}$, we get a simplicial complex. This complex is called the **increasing chain complex of the poset.** Note that this complex depends on the labeling.

Increasing Chain Complex

Lets look at an example. Consider the edge labeling of the lattice of flats of a graph G as follows. Order the edges of G lexicographically. Label the edge from H_1 to H_2 by $\min E(H_2) \setminus E(H_1)$. This labeling is known as the **minimum labeling** and was introduced by Björner.

Minimum Labeling Example



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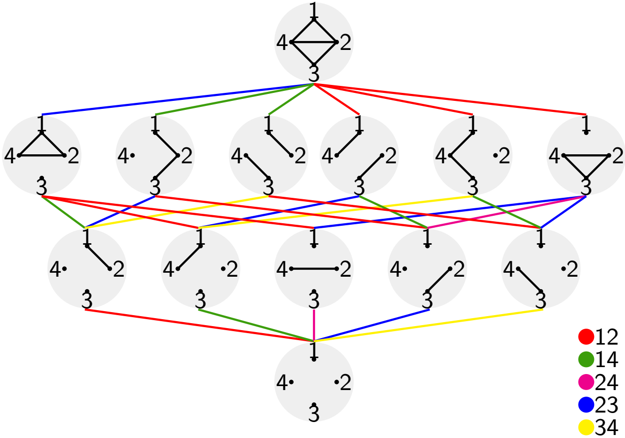

This labeling has the rank two switching property and the chains determine the elements.

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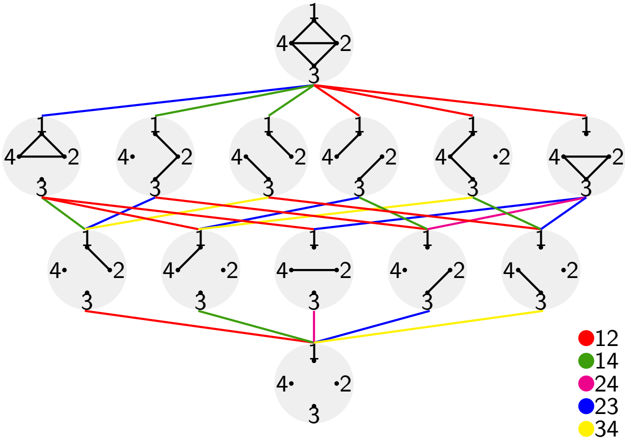
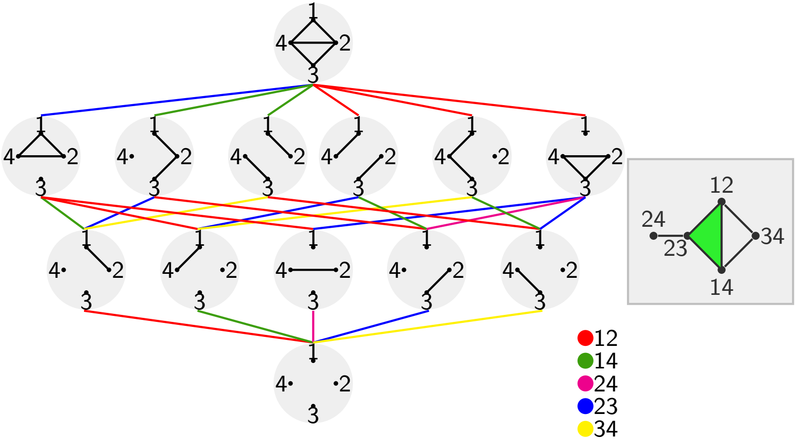


The sets of labels that appear along increasing chains are

$$\{12, 14, 23\}, \{12, 34\}, \{14, 34\}, \{23, 24\}$$

and their subsets.

Increasing Chain Complex Example



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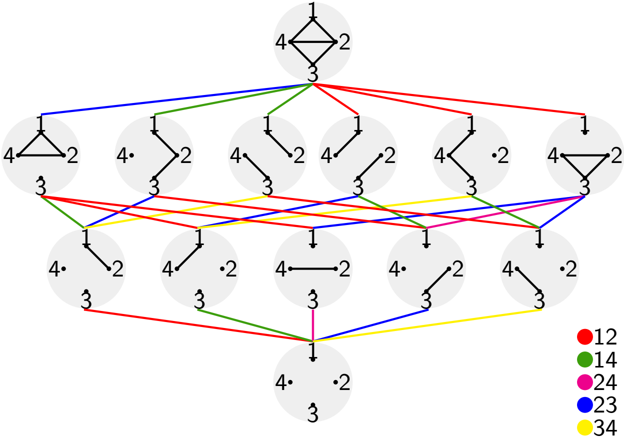

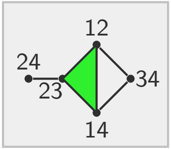

Let W_i be the number faces of size i in the increasing chain complex. Then

$$W_0 = 1, W_1 = 5, W_2 = 6, W_3 = 1$$

Increasing Chain Complex Properties

Proposition

Let P be a graded poset with an ER-labeling such that

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Then the f-vector of the increasing chain complex correspond to the Whitney numbers of the second kind of the poset.

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Proposition

Different labelings can produce non-isomorphic increasing chain complexes.

THANK YOU