Increasing Spanning Forests in Graphs

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joint work with

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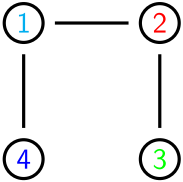
SIAM DM 2018

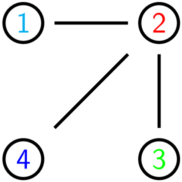
June 8, 2018

Increasing Trees

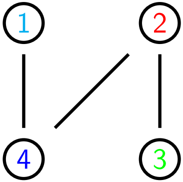
Let T be a tree with vertices labeled by distinct integers. We say T is *increasing* if the labels along any path starting at the smallest vertex are increasing.



Increasing



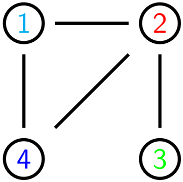
Increasing



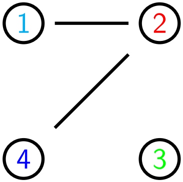
Not increasing

Increasing Spanning Forest

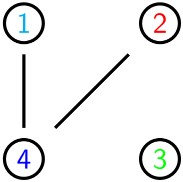
Let G be a graph with vertices labeled $1, 2, \ldots, n$. We say a spanning subgraph of G is an *increasing spanning forest of G* if each connected component of the forest is an increasing tree.



G

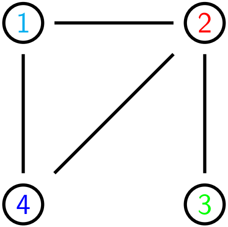


Increasing forest of G



Not an increasing forest of G

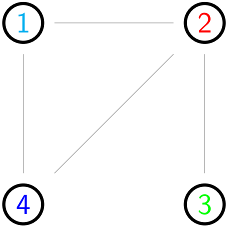
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	
1	
2	
3	

Counting Increasing Spanning Forests



G

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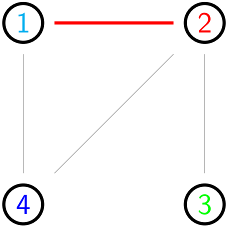
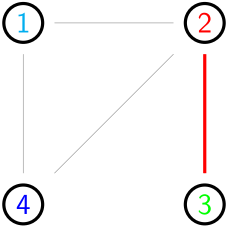
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	1
2	
3	

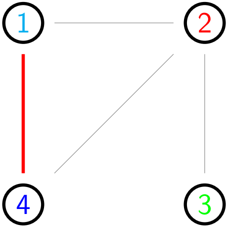
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	$1 + 1$
2	
3	

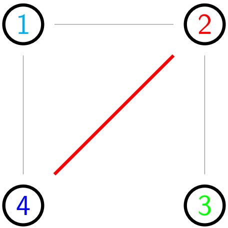
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	$1+1+1$
2	
3	

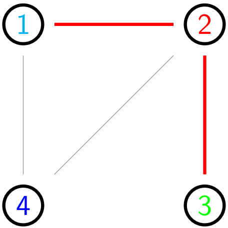
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	$1 + 1 + 1 + 1 = 4$
2	
3	

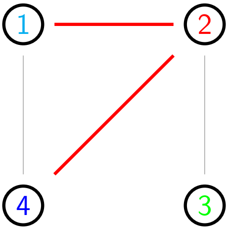
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	4
2	1
3	

Counting Increasing Spanning Forests



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Number of Edges	Number of ISFs
0	1
1	4
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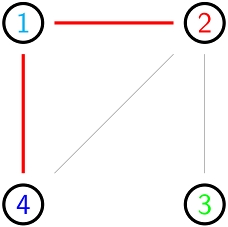
3	

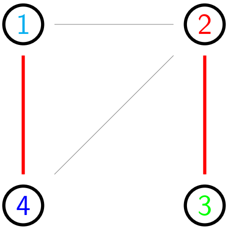
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	4
2	$1+1+1$
3	

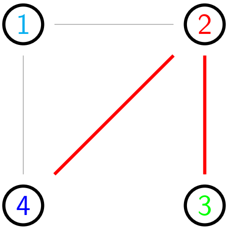
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	4
2	$1+1+1+1$
3	

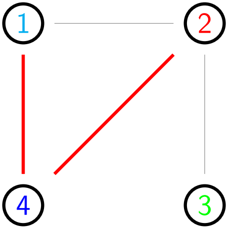
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	4
2	$1+1+1+1+1$
3	

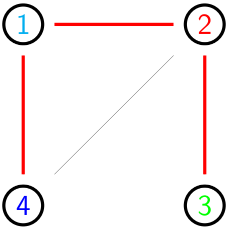
Counting Increasing Spanning Forests



Not an increasing forest

Number of Edges	Number of ISFs
0	1
1	4
2	5
3	

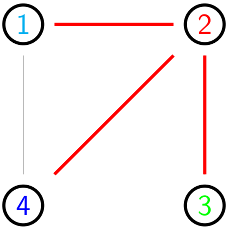
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	4
2	5
3	1

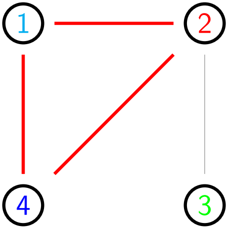
Counting Increasing Spanning Forests



G

Number of Edges	Number of ISFs
0	1
1	4
2	5
3	$1 + 1$

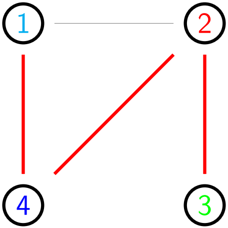
Counting Increasing Spanning Forests



Not an increasing forest

Number of Edges	Number of ISFs
0	1
1	4
2	5
3	2

Counting Increasing Spanning Forests



Not an increasing forest

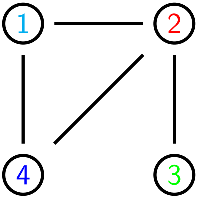
Number of Edges	Number of ISFs
0	1
1	4
2	5
3	2

ISF Generating Function

Let G be a graph with vertices labeled by $\{1, 2, \ldots, n\}$ and let $isf_k(G)$ be the number of increasing spanning forests of G with k edges. The *increasing spanning forest generating function* is given by

$$ISF(G, t) = \sum_{k=0}^{n-1} isf_k(G) t^{n-k}.$$

Example



G

Number of Edges	Number of ISFs
0	1
1	4
2	5
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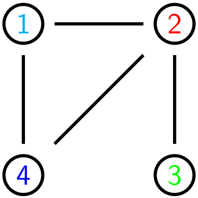
Using the table we get,

$$ISF(G, t) = t^4 + 4t^3 + 5t^2 + 2t$$

Example



G

Number of Edges	Number of ISFs
0	1
1	4
2	5
3	2

Using the table we get,

$$ISF(G,t) = t^4 + 4t^3 + 5t^2 + 2t = t(t+1)^2(t+2).$$

Partition of the Edge Set

Let G be a graph with vertices labeled by $\{1, 2, \ldots, n\}$. For each $1 \leq j \leq n$, let E_j be the set of edges of G of the form ij where $i < j$

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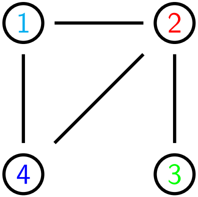

G

For G we get the following sets

$$E_1 =$$

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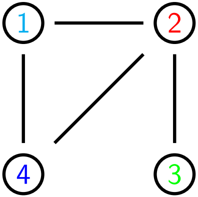

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$$E_1 = \emptyset, E_2 =$$

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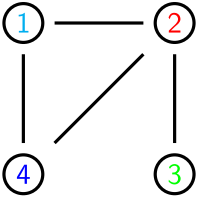

G

For G we get the following sets

$$E_1 = \emptyset, E_2 = \{12\}, E_3 =$$

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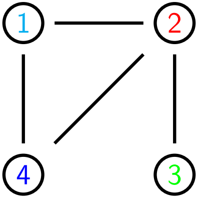

G

For G we get the following sets

$$E_1 = \emptyset, E_2 = \{12\}, E_3 = \{23\}, E_4 =$$

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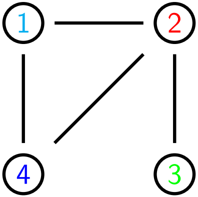

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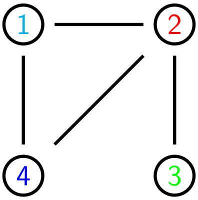

G

For G we get the following sets

$$E_1 = \emptyset, E_2 = \{12\}, E_3 = \{23\}, E_4 = \{14, 24\}.$$

And the increasing spanning forest generating function is given by

$ISF(G, t) = t(t+1)^2(t+2) = (t + |E_1|)(t + |E_2|)(t + |E_3|)(t + |E_4|)$,

where $|E_i|$ is the number of edges in E_i.

Increasing Spanning Forests

Theorem (H-Sagan)

Let G be a graph with vertices labeled by $\{1, 2, \ldots, n\}$. Define the E_j as before. Then

$$ISF(G, t) = (t + |E_1|)(t + |E_2|) \cdots (t + |E_n|).$$

Increasing Spanning Forests

Theorem (H-Sagan)

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$$ISF(G, t) = (t + |E_1|)(t + |E_2|) \cdots (t + |E_n|).$$

A note of caution: $ISF(G, t)$ does depend on the choice of labeling of the vertex set.

Graph Coloring

Let G be a graph, a *proper coloring* of G is a choice of color for each vertex in G such that any two vertices which have an edge between them are colored differently.

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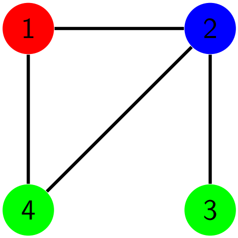
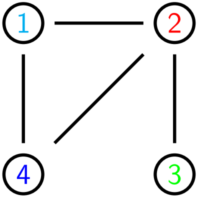


A proper coloring using three colors

The Chromatic Polynomial

Let G be a graph the *chromatic polynomial* of G is

$$\chi(G,t) = \text{ the number of proper colorings of } G \text{ with } t \text{ colors.}$$

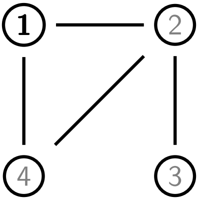


G

The Chromatic Polynomial

Theorem

The chromatic polynomial is a polynomial.



G

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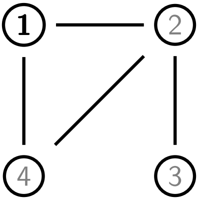

G

$\chi(G, t) = t$

The Chromatic Polynomial

Theorem

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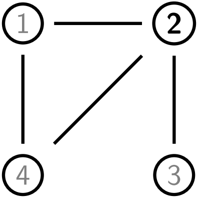


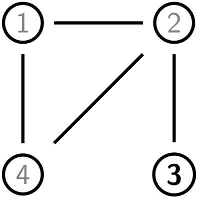
G

$\chi(G, t) = t(t-1)$

The Chromatic Polynomial

Theorem

The chromatic polynomial is a polynomial.



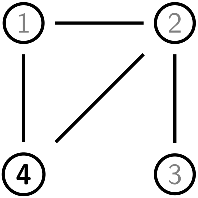
G

$\chi(G, t) = t(t-1)(t-1)$

The Chromatic Polynomial

Theorem

The chromatic polynomial is a polynomial.

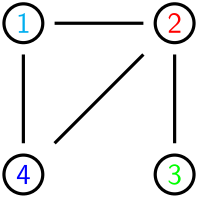


G

$$\chi(G, t) = t(t-1)(t-1)(t-2)$$

The Chromatic Polynomial and Increasing Forests

Let us again consider the graph below.

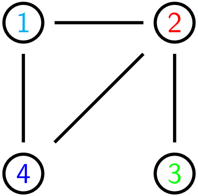


G

$$\chi(G,t) = t(t-1)^2(t-2).$$

The Chromatic Polynomial and Increasing Forests

Let us again consider the graph below.



G

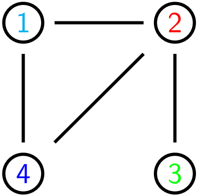
$$\chi(G,t) = t(t-1)^2(t-2).$$

Recall that for this labeling of the vertices,

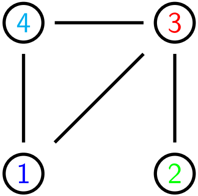
$$ISF(G,t) = t(t+1)^2(t+2).$$

Perfect Elimination Order

Let G be a graph with vertices labeled by $\{1, 2, \ldots, n\}$. We say this vertex labeling is a *perfect elimination order (PEO)* provided whenever ik, jk are edges with $i < j < k$, ij is an edge too.



A PEO



Not A PEO

The Chromatic Polynomial and Increasing Forests

Theorem (H-Sagan)

Let G be a graph with vertices labeled by $\{1, 2, \ldots, n\}$. Then

$$ISF(G, t) = (-1)^n \chi(G, -t)$$

if and only if the labeling on the vertices is a perfect elimination order.

Permutation Patterns

Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ and $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ be permutations. We say π *contains* σ if there exists a (not necessarily continuous) subsequence of π which is in the same relative order as σ, otherwise we say π *avoids* σ.

Example The permutation 6892 contains 231 from the 682 (or 692 or 892). The permutation 1324 avoids 231.

Permutation Patterns

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Example The permutation 6892 contains 231 from the 682 (or 692 or 892). The permutation 1324 avoids 231.
Note that a permutation is increasing if and only if it avoids 21.

Tight Trees

Let T be a tree with vertices labeled by distinct integers and let σ be a permutation. We say T *avoids* σ if for every path starting at the smallest vertex of T the permutation along the path avoids σ. Note that an increasing tree is just a tree which avoids 21.

We say a tree is *tight* if it avoids the patterns $231, 312$, and 321.

Tight Trees

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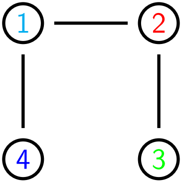
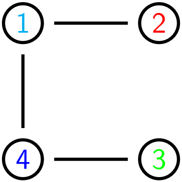
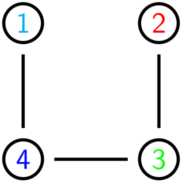


$123, 14$
Tight



$12, 143$
Tight



1432
Not Tight

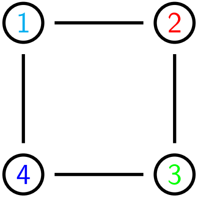
Tight Spanning Forests

Just as we did with increasing spanning forests, we create a generating function.

$$TSF(G, t) = \sum_{k=0}^{n-1} tsf_k(G) t^{n-k}$$

where G has n vertices and tsf_k is the number of spanning forests of G with k edges such that each component is a tight tree.

Example



G

Number of Edges	Number of TSFs
0	1
1	4
2	6
3	3

Using the table we get,

$$TSF(G, t) = t^4 + 4t^3 + 6t^2 + 3t$$

Tight Spanning Forests

Theorem (H-Martin-Sagan)

Let G be a graph. There is a labeling of G so that all the roots of $TSF(G, t)$ are integers if and only if G is a forest.

Tight Spanning Forests

Theorem (H-Martin-Sagan)

Let G be a graph. There is a labeling of G so that all the roots of $TSF(G, t)$ are integers if and only if G is a forest.

Martin, Sagan, and I also introduced a new type of labeling with the property that $TSF(G, t)$ is (up to a sign) the chromatic polynomial of G if and only if the G is labeled that way.

Thank You!