A Combinatorial Hopf Algebra of Simplicial Complexes

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Joint work with Carolina Benedetti and John Machacek

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Background

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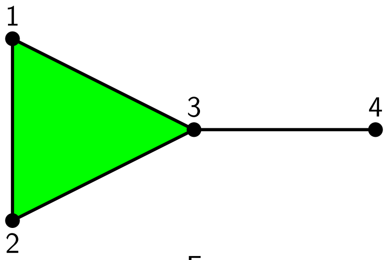
The elements of dimension 0 are called *vertices*. The elements of dimension 1 are called *edges*.

Example



Γ

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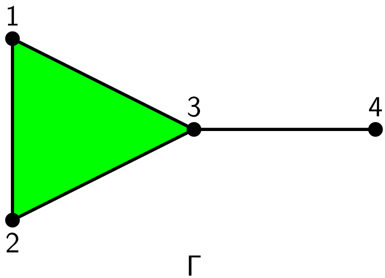

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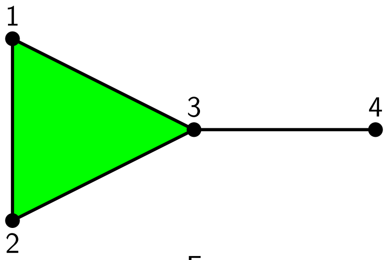

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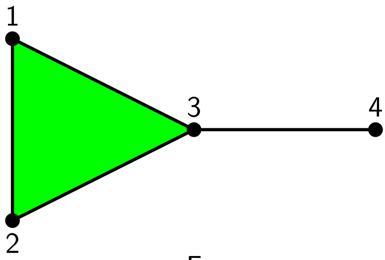

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The facets of Γ are $\{1, 2, 3\}$ and $\{3, 4\}$. The vertices are $1, 2, 3, 4$.
The dimension of Γ is 2.

Hopf Algebra Structure

Let $\mathbb{K}$ be a field and let

$$\mathcal{A} = \bigoplus_{n \geq 0} A_n$$

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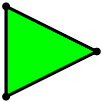
$\Gamma =$  $\quad \Theta = \bullet$

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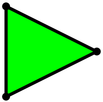
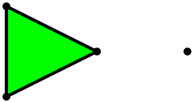
$\Gamma =$  $\Theta =$ 

then

$m(\Gamma \otimes \Theta) =$ 

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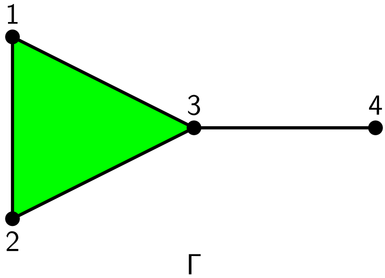

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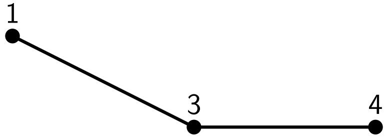


Γ_T

Hopf Algebra Structure

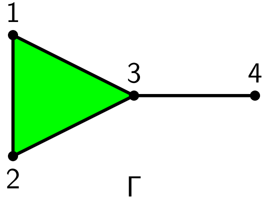
Coproduct

The coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is given by

$$\Delta(\Gamma) = \sum_{T \subseteq V(\Gamma)} \Gamma_T \otimes \Gamma_{V(\Gamma) \setminus T}$$

Coproduct Example

Suppose Γ was as before



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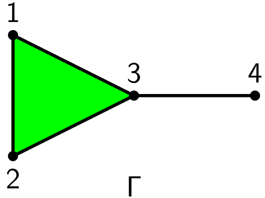

Γ

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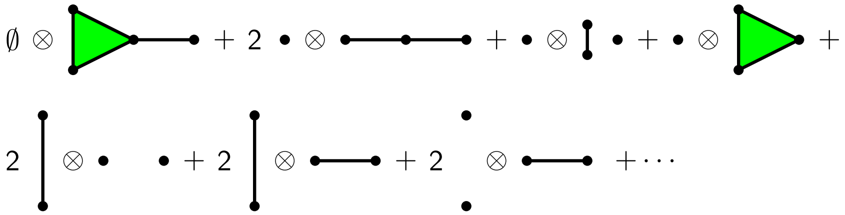

Antipode

Let Γ be a simplicial complex and let k be a nonnegative integer. The *k-skeleton* of Γ is

$$\Gamma^{(k)} = \{X \mid X \in \Gamma \text{ and } \dim(X) \leq k\}$$

Antipode

Let $G = (V, E)$ be a graph. A *flat* of G is a subset F of the edge set such that in the graph with vertex set V and edge set F, each connected component is an induced subgraph of G.

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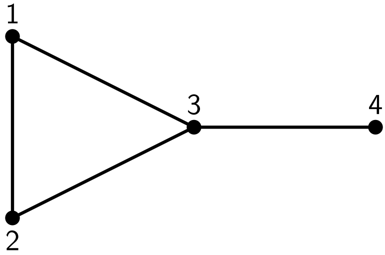

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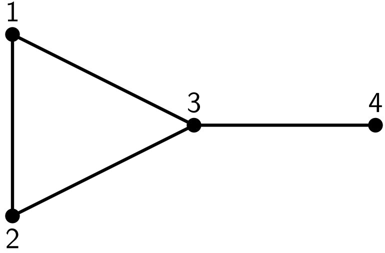

The flats are

$$\emptyset, \{12\}, \{13\}, \{23\}, \{34\}, \{12, 13, 23\},$$

$$\{12, 34\}, \{13, 34\}, \{23, 34\}, \{12, 13, 23, 34\}.$$

Antipode

Let Γ be a simplicial complex with vertex set V. Let F be a flat of the one-skeleton $\Gamma^{(1)}$. Define

$$\Gamma_F = \{X \in \Gamma \mid X^{(1)} \subseteq G_{V,F}\}$$

where $G_{V,F}$ is the subgraph of the one-skeleton of $\Gamma^{(1)}$ with vertex set V and edge set F.

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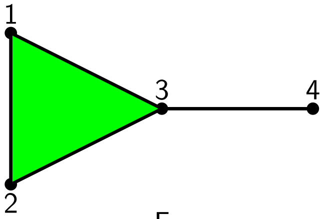
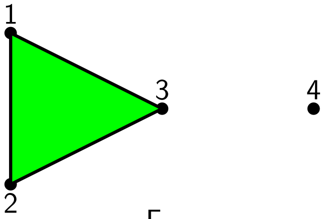
Example Let $F = \{12, 13, 23\}$. Then



Γ



Γ_F

Antipode

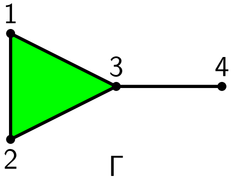
Theorem

Let Γ be a simplicial complex, then the antipode is given by

$$S(\Gamma) = \sum_F (-1)^{c(F)} a(\Gamma^{(1)}/F)\Gamma_F$$

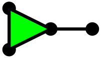
where the sum is over all flats of the one-skeleton of Γ, $c(F)$ is the number of connected components of $G_{V(\Gamma),F}$, and $a(\Gamma^{(1)}/F)$ is the number of acyclic orientations of $\Gamma^{(1)}/F$.

Antipode Example

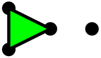
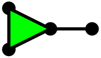


F a flat	$(-1)^{c(F)}$	$a(\Gamma^{(1)}/F)$
$\emptyset$	$(-1)^4$	12
$\{12\}$	$(-1)^3$	4
$\{13\}$	$(-1)^3$	4
$\{23\}$	$(-1)^3$	4
$\{34\}$	$(-1)^3$	6
$\{12, 34\}$	$(-1)^2$	2
$\{13, 34\}$	$(-1)^2$	2
$\{23, 34\}$	$(-1)^2$	2
$\{12, 13, 23\}$	$(-1)^2$	2
$\{12, 13, 23, 34\}$	$(-1)^1$	1

Antipode Example

$S($  $) =$

$12($  $) \; - \; 18($  $) \; + \; 2($  $) \; +$

$4($  $) \; + \; 2($  $) \; - \; 1($  $)$

Characters and Colorings

Given a simplicial complex Γ and a nonnegative integer s, an *s-simplicial coloring* of Γ is an assignment of colors to the vertices of Γ such that no s-dimensional face is monochromatic.

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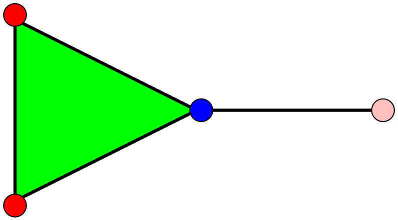

A coloring which is a 2-simplicial coloring, but not a 1-simplicial coloring

Characters and Colorings

Let Γ be a simplicial complex with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let s be a nonnegative integer. The *s-chromatic symmetric function* is

$$\Psi_s(\Gamma) = \sum_{f:V \to \mathbb{P}} x_{f(v_1)} x_{f(v_2)} \cdots x_{f(v_n)}$$

where the sum is over all s-simplicial colorings (with colors from the positive integers $\mathbb{P}$).

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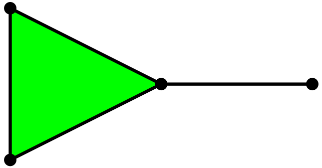

Γ

The terms $x_1^2 x_2^2$ and $x_1^3 x_2$ appear in $\Psi_2(\Gamma)$, but x_1^4 does not.

Characters and Colorings

For each nonnegative integer s, define

$$\zeta_s(\Gamma) = \begin{cases} 1 & \text{if } \dim(\Gamma) < s \\ 0 & \text{otherwise} \end{cases}$$

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The (quasi)symmetric function associated with ζ_s is the s-chromatic symmetric function.

Simplicial Chromatic Polynomial

Let Γ be a simplicial complex. The *s-chromatic polynomial* is

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Proposition

Let Γ be a simplicial complex. Then for all positive integers s, we have

$$\chi_s(\Gamma, -1) = \sum_{F; \dim(\Gamma_F) < s} (-1)^{c(F)} a(\Gamma^{(1)}/F)$$

where the sum is over all flats of the one-skeleton such that $dim(\Gamma_F) < s$.

q-analogs

For each nonnegative integer s, define

$$\zeta_{s,q}(\Gamma) = \begin{cases} q^{rk(\Gamma^{(1)})} & \text{if } \dim(\Gamma) < s \\ 0 & \text{otherwise} \end{cases}$$

and extend linearly. Here $rk(\Gamma^{(1)})$ is the rank of the one-skeleton of Γ.

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and extend linearly. Here $rk(\Gamma^{(1)})$ is the rank of the one-skeleton of Γ.

Note that $\zeta_{s,q}(\Gamma) = q^{rk(\Gamma^{(1)})}\zeta_s(\Gamma)$. Moreover, note that

$$\zeta_{s,1} = \zeta_s$$

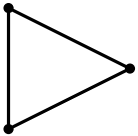
and

$$\zeta_{s,0} = \zeta_1$$

provided $s \geq 1$.

q-analogs
Example

Suppose Γ is the complete graph on 3 vertices.



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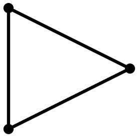

K_3

Then for $s > 1$,

$$\Psi_{s,q}(K_3) = q^2 m_{(3)} + 3q m_{(2,1)} + 6 m_{(1,1,1)}.$$

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$$-q^2 - 6q - 6 = -(q^2 + 6q + 6).$$

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Recall that the Eulerian polynomial is given by

$$A_n(q+1) = \sum_{\omega \in \mathfrak{S}_n} (q+1)^{des\ \omega}.$$

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For $n = 3$, we get

$$A_3(q+1) = q^2 + 6q + 6.$$

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Let $\Psi_{s,q}$ be the symmetric function associated with $\zeta_{s,q}$. Let $ps^1(\Psi_{s,q})(-1)$ be the principal specialization of $\Psi_{s,q}$ at -1.

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Proposition

Let K_n be the complete graph on n vertices. Then

$$ps^1(\Psi_{s,-q}(K_n))(-1) = (-1)^n A_n(q+1).$$

where

$$A_n(q+1) = \sum_{\omega \in \mathfrak{S}_n} (q+1)^{des\ \omega}.$$

is the Eulerian polynomial.

q-analogs

By calculating the principle specialization a different way, one obtains the following identity

$$A_n(q+1) = \sum_{\alpha \vDash n} \binom{n}{\alpha} q^{n-l(\alpha)}.$$

Here the sum is over compositions of n, $\binom{n}{\alpha} = \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_l!}$, and $l(\alpha)$ is the length of the composition α.

q-analogs

How well can $\Psi_{s,q}$ distinguish graphs? Since the chromatic symmetric function can be recovered from $\Psi_{s,q}$ by setting $q = 0$, anything the chromatic symmetric function can tell apart, so can this q-analog.

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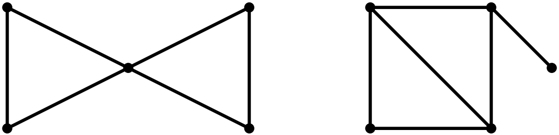

The chromatic symmetric function cannot distinguish the graphs above, but the q-analog can. The coefficient of $m_{(3,2)}$ for the graph on the left is $2q^3 + 8q^2$ whereas the coefficient for the graph on the right is $3q^3 + 6q^2 + q$.

Thank You!