Dual Immaculate Quasisymmetric Functions Expand Positively into Young Quasisymmetric Schur Functions

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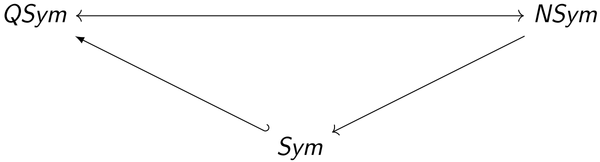
Some Schur-like Bases of *QSym* and *NSym*



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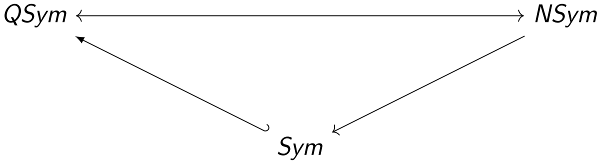
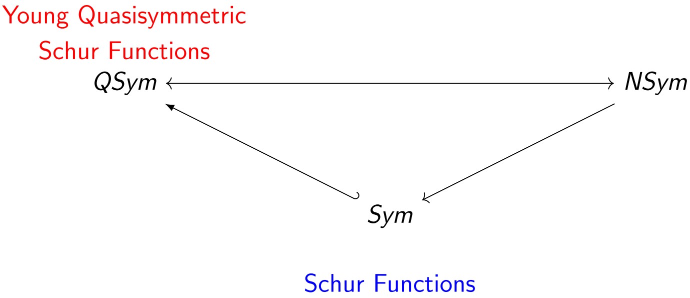


Schur Functions

Some Schur-like Bases of *QSym* and *NSym*



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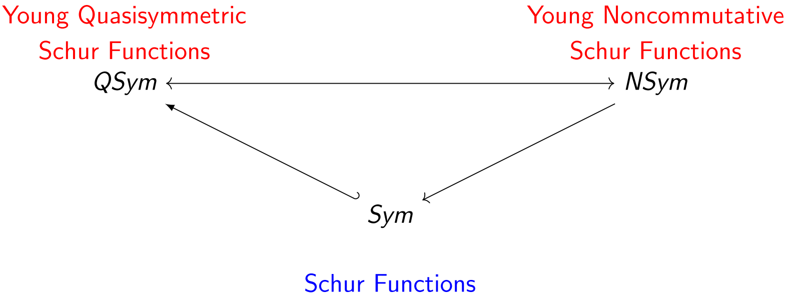

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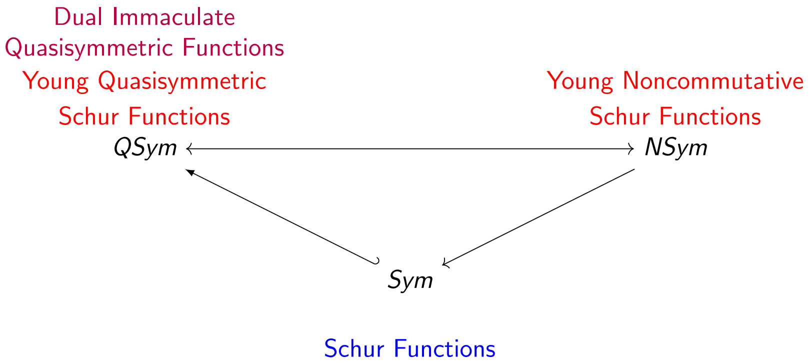

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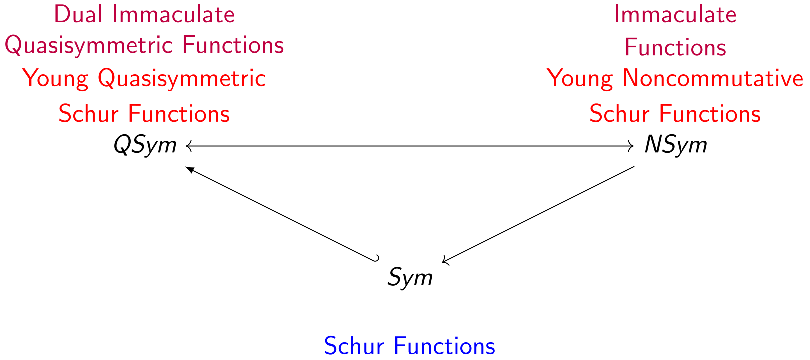


Some Schur-like Bases of *QSym* and *NSym*

QSym:
Dual Immaculate Quasisymmetric functions:$\{\mathfrak{d}I_\alpha\}_\alpha$
(Berg, Bergeron, Saliola, Serrano, Zabrocki, 2012)

- Generated by immaculate tableaux.
- Decompose positively into the fundamental basis.

Young Quasisymmetric Schur functions:$\{YQS_\alpha\}_\alpha$
(Luoto, Mykytiuk, and van Willigenburg, 2013)

- Generated by Young composition tableaux.
- Decompose positively into the fundamental basis.

Question: How are these bases related?

Some Schur-like Bases of *QSym* and *NSym*

NSym:

Immaculate Functions $\{I_\alpha\}_\alpha$

- Can be defined using noncommutative Bernstein operators.
- Can be defined using a noncommutative analogue of the Jacobi-Trudi identity.

Young noncommutative Schur functions $\{\hat{\mathbf{s}}_\alpha\}_\alpha$

- Littlewood-Richardson rule,

$$\hat{\mathbf{s}}_\alpha \hat{\mathbf{s}}_\beta = \sum C^{\gamma}_{\alpha,\beta} \hat{\mathbf{s}}_\gamma$$

  where $C^{\gamma}_{\alpha,\beta}$ counts certain tableaux.

Question: How are these bases related?

Main Theorem

Theorem

Let dI_α be the dual immaculate function indexed by α and let YQS_β be the Young quasisymmetric Schur function indexed by β. Then

$$dI_\alpha = \sum_\beta c_{\alpha,\beta} YQS_\beta$$

where $c_{\alpha,\beta}$ is the number of DIRTs of shape β and row strip shape α^{rev}.

We use an insertion algorithm to prove this result.

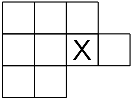
Compositions

To each subset of $[n-1]$, we associate a composition by sending the set $\{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$ to the composition $(a_1, a_2 - a_1, \ldots, a_k - a_{k-1}, n - a_k)$

For example, if $n = 4$,

$$comp(\{1, 3\}) = (1, 2, 1).$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vDash n$. The *Young composition diagram* of shape α is the collection of left-justified cells such that row i has α_i cells. We will use French notation. The diagram below has shape $(2, 4, 3)$ and the cell with an X is in position $(3, 2)$.



Standard Immaculate Tableaux

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vDash n$. A filling of the Young composition diagram of shape α with the numbers $1, 2, \ldots, n$ is called a *standard immaculate tableau* if

1. The leftmost column is decreasing from top to bottom and
2. the rows are increasing from left to right.

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1. The leftmost column is decreasing from top to bottom and
2. the rows are increasing from left to right.

Example

3	4	5	6
1	2		

2	4	5	6
1	3		

2	3	5	6
1	4		

2	3	4	6
1	5		

2	3	4	5
1	6		

The five standard immaculate tableaux of shape $(2, 4)$.

Dual Immaculate Quasisymmetric Functions

The *descent set* of a standard immaculate tableau T is given by

$$\mathrm{Des}_{dI}(T) = \{i \mid i+1 \text{ is in a row strictly above } i \text{ in } T\}$$

3	4	5	6
1	**2**		

2	4	5	6
1	**3**		

2	3	5	6
1	**4**		

2	3	4	6
1	**5**		

2	3	4	5
1	6		

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2	4	5	6
1	**3**		

2	3	5	6
1	**4**		

2	3	4	6
1	**5**		

2	3	4	5
1	6		

For a composition $\alpha \vDash n$, the dual immaculate quasisymmetric function dI_α expands into the fundamental basis as

$$dI_\alpha = \sum_T F_{comp(Des_{dI}(T))}$$

where the sum is over all standard immaculate tableaux of shape α.
So

$$dI_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)}.$$

Standard Young Composition Tableaux

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vDash n$. A filling of the Young composition diagram of shape α with the numbers $1, 2, \ldots, n$ is called a *standard Young composition tableau* if

1. The leftmost column is decreasing from top to bottom and
2. the rows are increasing from left to right and
3. (YCT triple rule) for every subarray as below, if $a > b$ then $a > c$, where if c is empty, $c = \infty$.

b	c
	a

Standard Young Composition Tableaux

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b	c
	a

Note that every standard Young composition tableau is a standard immaculate tableau.

3	4	5	6
1	2		

2	4	5	6
1	3		

2	3	5	6
1	4		

2	3	4	6
1	5		

2	3	4	5
1	6		

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b	c
	a

Note that every standard Young composition tableau is a standard immaculate tableau.

3	4	5	6
1	2		

2	**4**	5	6
1	**3**		

2	3	5	6
1	4		

2	3	4	6
1	5		

2	3	4	5
1	6		

All except the second tableau are a standard Young composition tableaux.

Young Quasisymmetric Schur Function

The *descent set* of a standard Young Composition Tableau T is given by

$$\mathrm{Des}_{YQS}(T) = \{i \mid i+1 \text{ is weakly left of } i \text{ in } T\}.$$

3	4	5	6
1	**2**		

2	**3**	5	6
1	4		

2	3	**4**	6
1	5		

2	3	4	**5**
1	6		

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3	4	5	6
1	**2**		

2	**3**	5	6
1	4		

2	3	**4**	6
1	5		

2	3	4	**5**
1	6		

It is important to note that the two definitions of descent set for the same filling do not necessarily agree.

3	4	5	6
1	**2**		

2	3	5	6
1	**4**		

2	3	4	6
1	**5**		

2	3	4	5
1	6		

Young Quasisymmetric Schur Function

For a composition $\alpha \vDash n$, the Young quasisymmetric Schur function YQS_α expands into the fundamental basis as

$$YQS_\alpha = \sum_T F_{comp(Des_{YQS}(T))}$$

where the sum is over all standard Young composition tableaux of shape α.

Young Quasisymmetric Schur Function

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where the sum is over all standard Young composition tableaux of shape α.

3	4	5	6
1	**2**		

2	**3**	5	6
1	4		

2	3	**4**	6
1	5		

2	3	4	**5**
1	6		

So

$$YQS_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)}$$

Insertion Algorithm

Given a standard immaculate tableau, we obtain its reading word by reading left to right starting in the top row and working down.

We then insert this word to get a pair (P, Q) where P is a standard Young composition tableau and Q is a recording tableau.

Insertion Example

2	3	5	6
1	**4**		

$\longrightarrow$ 2356**14** $\longrightarrow$ $\left(\qquad , \qquad \right)$

Insertion Example

$$\begin{array}{|c|c|c|c|}\hline 2 & 3 & 5 & 6 \\ \hline \mathbf{1} & \mathbf{4} \\ \cline{1-2}\end{array} \longrightarrow 2356\mathbf{14} \longrightarrow \left(\boxed{2} \quad , \quad \boxed{1} \right)$$

We insert the 2 into an empty filling.

Insertion Example

$$\begin{array}{|c|c|c|c|}\hline 2 & 3 & 5 & 6 \\ \hline \mathbf{1} & \mathbf{4} \\ \cline{1-2} \end{array} \longrightarrow 235614 \longrightarrow \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \right)$$

The 3 is placed next to the 2.

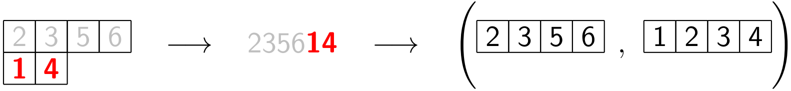
Insertion Example

2	3	5	6
1	**4**		

$\longrightarrow$ 235614 $\longrightarrow$ $\left(\begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline \end{array} \;,\; \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right)$

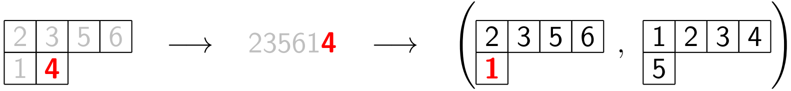
The 5 is placed next to the 3.

Insertion Example



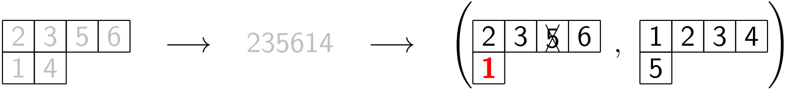
The 6 is placed next to the 5.

Insertion Example



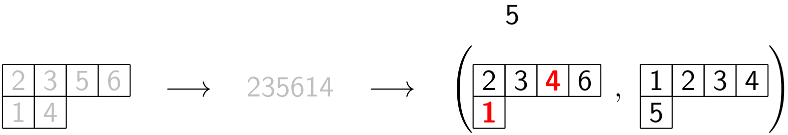
Since 1 is smaller than all other elements, it starts a new row.

Insertion Example



The 4 bumps the 5 and the insertion continues with the 5.

Insertion Example



The 4 bumps the 5 and the insertion continues with the 5.

Insertion Example

$$\begin{array}{|c|c|c|c|}\hline 2&3&5&6\\\hline 1&4\\\cline{1-2}\end{array} \quad\longrightarrow\quad 235614 \quad\longrightarrow\quad \left(\begin{array}{|c|c|c|c|}\hline 2&3&\mathbf{4}&6\\\hline \mathbf{1}&5\\\cline{1-2}\end{array}\ ,\ \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\right)$$

The 5 is placed next to the 1.

Insertion Example

$$\begin{array}{|c|c|c|c|}\hline 2 & 3 & 5 & 6 \\ \hline \mathbf{1} & \mathbf{4} \\ \cline{1-2}\end{array} \longrightarrow 2356\mathbf{1}\mathbf{4} \longrightarrow \left(\begin{array}{|c|c|c|c|}\hline 2 & 3 & \mathbf{4} & 6 \\ \hline \mathbf{1} & 5 \\ \cline{1-2}\end{array} \, , \, \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array} \right)$$

$\mathrm{Des}_{dI}(T) = \{i \mid i+1$ is in a row strictly above i in $T\}$

$\mathrm{Des}_{YQS}(T) = \{i \mid i+1$ is weakly left of i in $T\}$

Insertion Example

Immaculate Tableaux	Young Composition Tableaux	Recording Tableaux
3 4 5 6 1 **2**	3 4 5 6 1 **2**	1 2 3 4 5 6
2 4 5 6 **1** **3**	2 **3** 5 6 **1** 4	1 2 3 4 5 6
2 3 5 6 **1** **4**	2 3 **4** 6 **1** 5	1 2 3 4 5 6
2 3 4 6 **1** **5**	2 3 4 **5** **1** 6	1 2 3 4 5 6
2 3 4 5 **1** 6	2 3 4 5 6 **1**	1 2 3 4 6 5

Insertion Example

Immaculate Tableaux	Young Composition Tableaux	Recording Tableaux
3 4 5 6 1 **2**	3 4 5 6 1 2	1 2 3 4 5 6
2 4 5 6 **1** **3**	2 3 5 6 1 4	1 2 3 4 5 6
2 3 5 6 **1** **4**	2 3 4 6 1 5	1 2 3 4 5 6
2 3 4 6 **1** **5**	2 3 4 5 1 6	1 2 3 4 5 6
2 3 4 5 **1** 6	2 3 4 5 6 1	1 2 3 4 6 5

Insertion Example

Immaculate Tableaux

3	4	5	6
1	**2**		

2	4	5	6
1	**3**		

2	3	5	6
1	**4**		

2	3	4	6
1	**5**		

2	3	4	5
1	6		

Young Composition Tableaux

3	4	5	6
1	2		

2	3	5	6
1	4		

2	3	4	6
1	5		

2	3	4	5
1	6		

2	3	4	5	6
1				

Recording Tableaux

1	2	3	4
5	6		

1	2	3	4
5	6		

1	2	3	4
5	6		

1	2	3	4
5	6		

1	2	3	4	6
5				

$$dI_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)}$$

Insertion Example

Immaculate Tableaux

3	4	5	6
1	2		

2	4	5	6
1	3		

2	3	5	6
1	4		

2	3	4	6
1	5		

2	3	4	5
1	6		

Young Composition Tableaux

3	**4**	**5**	**6**
1	**2**		

2	**3**	**5**	**6**
1	**4**		

2	**3**	**4**	**6**
1	**5**		

2	**3**	**4**	**5**
1	**6**		

2	3	4	5	6
1				

Recording Tableaux

1	2	3	4
5	6		

1	2	3	4
5	6		

1	2	3	4
5	6		

1	2	3	4
5	6		

1	2	3	4	6
5				

$$dI_{(2,4)} = \boldsymbol{F_{(2,4)}} + \boldsymbol{F_{(1,2,3)}} + \boldsymbol{F_{(1,3,2)}} + \boldsymbol{F_{(1,4,1)}} + F_{(1,5)}$$

$$= YQS_{(2,4)} +$$

Insertion Example

Immaculate Tableaux	Young Composition Tableaux	Recording Tableaux
3 4 5 6 1 2	3 4 5 6 1 2	1 2 3 4 5 6
2 4 5 6 1 3	2 3 5 6 1 4	1 2 3 4 5 6
2 3 5 6 1 4	2 3 4 6 1 5	1 2 3 4 5 6
2 3 4 6 1 5	2 3 4 5 1 6	1 2 3 4 5 6
2 3 4 5 1 6	**2 3 4 5 6** 1	1 2 3 4 6 5

$$dI_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + \boldsymbol{F_{(1,5)}}$$
$$= YQS_{(2,4)} + YQS_{(1,5)}.$$

Insertion Example

Immaculate Tableaux	Young Composition Tableaux	Recording Tableaux
3 4 5 6 / 1 **2**	3 4 5 6 / 1 **2**	1 2 3 4 / 5 6
2 4 5 6 / **1** **3**	2 **3** 5 6 / **1** 4	1 2 3 4 / 5 6
2 3 5 6 / **1** **4**	2 3 **4** 6 / **1** 5	1 2 3 4 / 5 6
2 3 4 6 / **1** **5**	2 3 4 **5** / **1** 6	1 2 3 4 / 5 6
2 3 4 5 / **1** 6	2 3 4 5 6 / **1**	1 2 3 4 6 / 5

$$dI_{(2,4)} = F_{(2,4)} + F_{(1,2,3)} + F_{(1,3,2)} + F_{(1,4,1)} + F_{(1,5)}$$
$$= YQS_{(2,4)} + YQS_{(1,5)}.$$

Row Strips and Row Strip Shapes

The recording tableaux obtained by inserting the reading word of an immaculate tableau has a certain form. To describe the properties, we need two definitions.

Let Q be a filling of a Young composition diagram with distinct entries $\{1, 2, \ldots, n\}$. A *row strip of length k* is a maximal sequence of k consecutive integers such that for all $1 \leq i < k$ in the row strip, $i + 1$ appears strictly right of i in Q.

The *row strip shape* is the composition $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ where α_i is the length of the row strip which starts with the number $\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + 1$

Example

The filling below has row strip shape $(2, 4, 3)$

1	2	8		
3	4	5	6	9
7				

Dual Immaculate Recording Tableau (DIRT)

If Q is a recording tableau obtained from inserting the reading word of a standard immaculate tableau of shape α then the following hold for Q.

1. The rows of Q increase from left to right.
2. The leftmost column of Q increases from top to bottom
3. The row strips start in the leftmost column of Q and the row strip shape is α^{rev}.
4. (Recording Triple Rule) Whenever the following subarray appears in Q with $a > b$, we also have $a > c$ where if c is empty, $c = \infty$.

a	
b	c

Such tableaux are called *dual immaculate recording tableaux (DIRT)*.

Main Theorem

Insertion gives a descent preserving bijection between standard immaculate tableaux of shape α and pairs (P, Q) where

- P is standard Young composition tableau.
- Q is a DIRT with row strip shape α^{rev}.
- P and Q have the same shape.

Main Theorem

Insertion gives a descent preserving bijection between standard immaculate tableaux of shape α and pairs (P, Q) where

- P is standard Young composition tableau.
- Q is a DIRT with row strip shape α^{rev}.
- P and Q have the same shape.

Theorem

Let dI_α be the dual immaculate function indexed by α and let YQS_β be the Young quasisymmetric Schur function indexed by β. Then

$$dI_\alpha = \sum_\beta c_{\alpha,\beta} YQS_\beta$$

where $c_{\alpha,\beta}$ is the number of DIRTs of shape β and row strip shape α^{rev}.

An algorithm to find DIRTs with fixed row strip shape

We can build a rooted tree to find all the DIRTs of row strip shape $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. For $1 \leq m \leq k$, the nodes of the tree at level m are DIRTs of row strip shape $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.

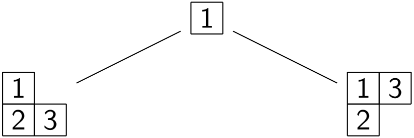
Finding DIRTs of row strip shape $(1, 2, 3)$

DIRTs of row strip shape (1, ,)

$\boxed{1}$

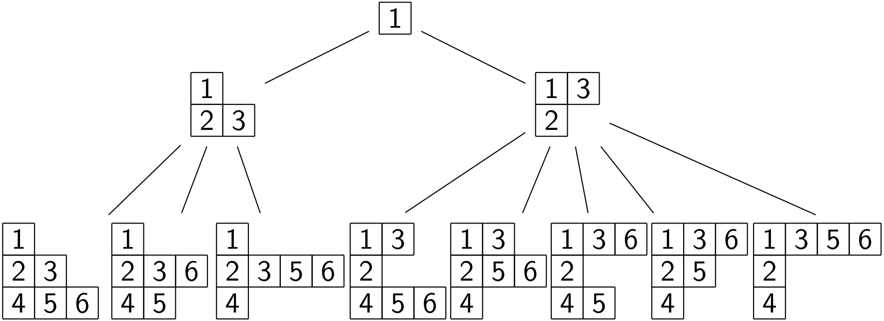
Finding DIRTs of row strip shape $(1, 2, 3)$

DIRTs of row strip shape $(1, 2, \ \ )$



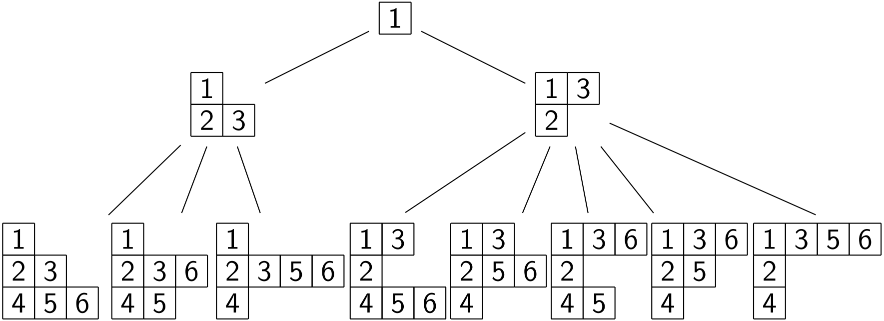
Finding DIRTs of row strip shape $(1, 2, 3)$

DIRTs of row strip shape $(1, 2, 3)$



Finding DIRTs of row strip shape $(1,2,3)$

DIRTs of row strip shape $(1,2,3)$



$$dI_{(3,2,1)} = YQS_{(3,2,1)} + YQS_{(2,3,1)} + YQS_{(1,4,1)} + YQS_{(3,1,2)} + YQS_{(1,3,2)} + YQS_{(2,1,3)} + YQS_{(1,2,3)} + YQS_{(1,1,4)}.$$

Decompositions in *NSym*

Theorem

Let $\hat{\mathbf{s}}_\alpha$ *be the Young noncommutative Schur function indexed by* α *and let* $\mathfrak{I}_\beta$ *be the immaculate function indexed by* β*. Then*

$$\hat{\mathbf{s}}_\alpha = \sum_\beta c_{\beta,\alpha} \mathfrak{I}_\beta$$

where $c_{\beta,\alpha}$ *is the number of DIRTs of shape* α *and row strip shape* β^{rev}*.*

Decompositions in *NSym*

Theorem

Let $\hat{\mathbf{s}}_\alpha$ be the Young noncommutative Schur function indexed by α and let I_β be the immaculate function indexed by β. Then

$$\hat{\mathbf{s}}_\alpha = \sum_\beta c_{\beta,\alpha} I_\beta$$

where $c_{\beta,\alpha}$ is the number of DIRTs of shape α and row strip shape β^{rev}.

Corollary

We have the following.

1. $I_\alpha = \hat{\mathbf{s}}_\alpha$ *if and only if α is a partition.*
2. *For the hook shape $(1^k, n-k)$, we have*

$$\hat{\mathbf{s}}_{(1^k,n-k)} = \sum_{\substack{\beta \vDash n \\ \ell(\beta)=k+1}} I_\beta .$$

Conjectures

Conjecture

Let $\alpha \vDash n$. *If*

$$YQS_\alpha = \sum b_{\alpha,\beta} dI_\beta$$

then $b_{\alpha,\beta} \in \{-1, 0, 1\}$. *Moreover, for a fixed* α*,*

$$\sum_\beta b_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = (1^k, n-k), \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture

Let λ *be a partition of* n *with* k *parts all of which are distinct. Then*

$$YQS_\lambda = \sum_{\pi \in S_k} (-1)^{\ell(\pi)} dI_{\pi(\lambda)}$$

where $\ell(\pi)$ *is the length of the permutation* π *and* $\pi(\lambda) = (\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(k)})$.

THANK YOU!