Counting the Maximal Chains of Bond Lattices

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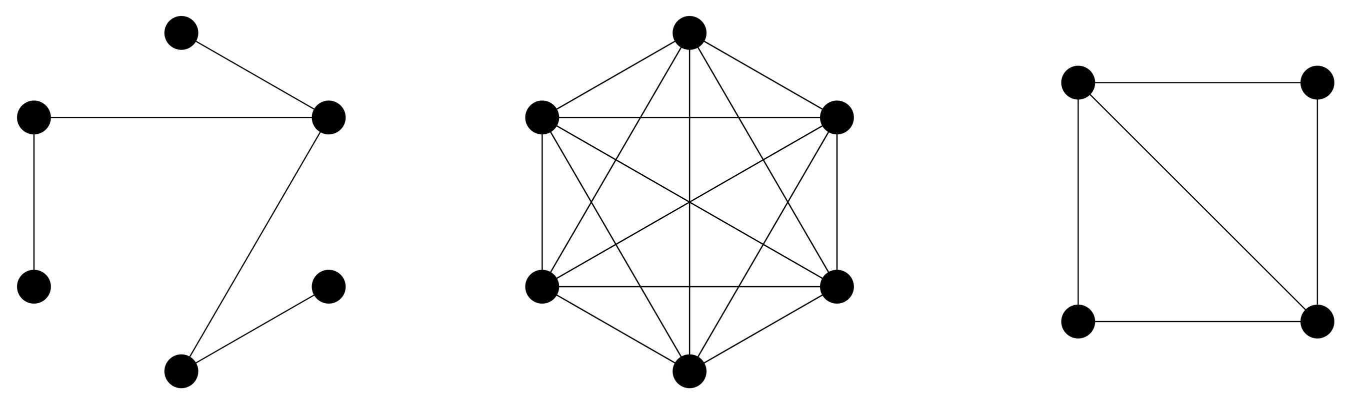
Graphs and Graph Theory

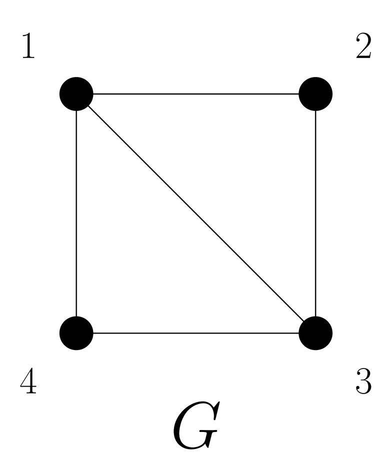
A **graph** is a set of points, called nodes or vertices, that are connected by lines, called edges. Graph theory, the study of these graphs, is an important tool in many fields of mathematics. Graphs are used to describe social networks, airplane flights, machine learning, and more.

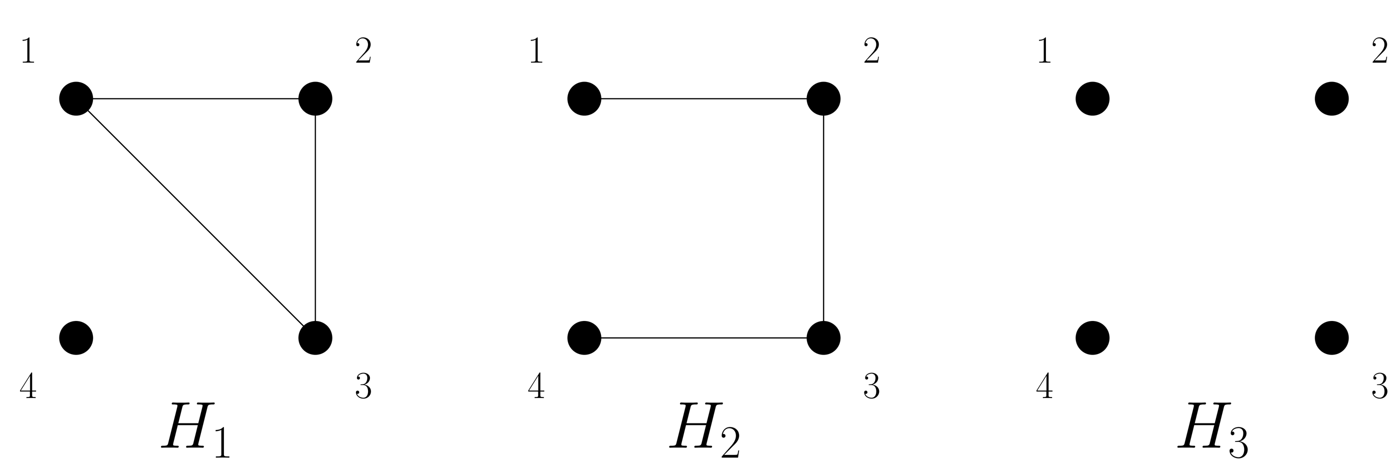


Some examples of graphs

What is a Bond?

A **subgraph** is a "piece" of a graph consisting of some of its vertices and edges.





A graph and some of its subgraphs

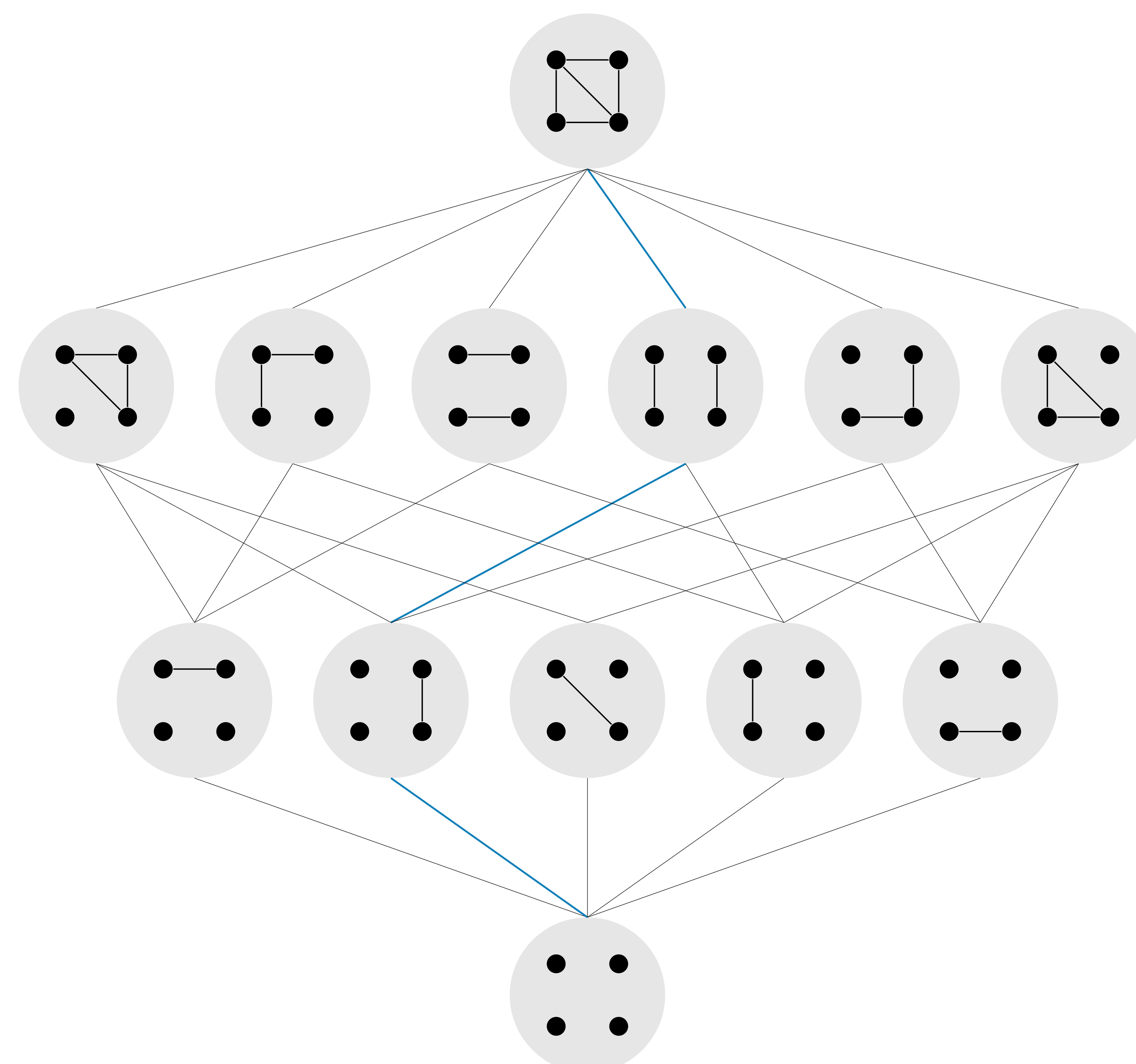
Let G be a graph. A subgraph H is a **bond** of G if it has the following two properties:

1. H contains all the vertices of G.
2. All of connected parts of H, or "islands", are **induced**. This means that if u and v are vertices in the same connected part of H and there is an edge connecting u and v in G, then there is an edge connecting u and v in H.

Try it! Take a look at H_1, H_2, and H_3 above. Which ones do you think are bonds of G? H_1 and H_3 are bonds, while H_2 is not.

What Is a Bond Lattice?

All of the bonds from a graph form a **partially ordered set**, or poset. A bond Y is higher in the order than another bond X if X is a subgraph of Y. The order is "partial" because not all of the bonds can be compared to one another. This ordering is called the **bond lattice** of the graph.



The bond lattice of the graph G to the left. One maximal chain is shown in blue.

Bond lattices encode useful combinatorial properties of the graphs they come from. One important attribute of a bond lattice is a **maximal chain**, which is any path from the bottom to the top of the lattice. For example, the above bond lattice has 14 maximal chains. Knowing the number of maximal chains can be helpful when comparing different bond lattices.

Research Questions

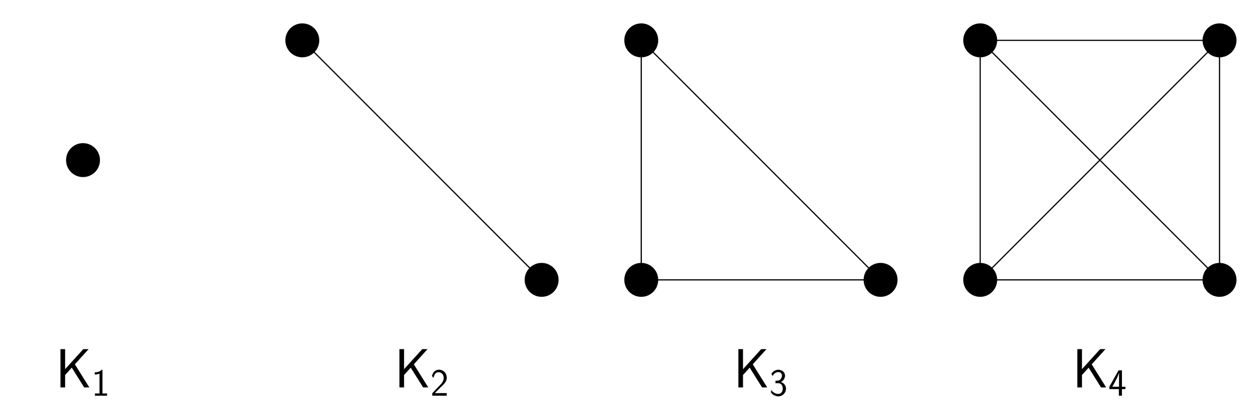
We looked at the following two questions.

1. Given a graph G, how many maximal chains are in the bond lattice of G?
2. Do there exist two bond lattices that are structurally different (i.e. not isomorphic) with the same number of maximal chains?

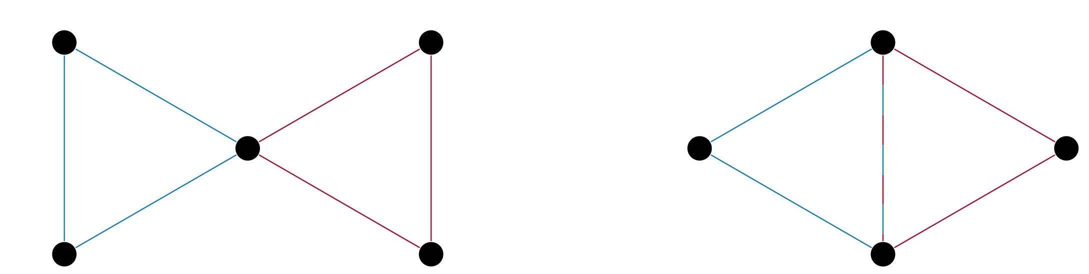
We examined special cases for the first question and took a computational approach to the second question.

Complete Graphs and Bond Lattices

A **complete graph** is a special type of graph where every pair of vertices has an edge between them. The complete graph on n vertices is denoted by K_n. Below are the complete graphs on one, two, three, and four vertices.



To get new graphs, we can "glue" copies of K_n together along smaller complete graphs.



Two K_3's glued along a K_1. $G_{3,1,2}$ Two K_3's glued along a K_2. $G_{3,2,2}$

We use the notation $G_{a,b,c}$ for the graph obtained by gluing c copies of K_a along the same K_b. So, the graphs above are $G_{3,1,2}$ and $G_{3,2,2}$.

Our goal was to see if we could find a pattern in the bond lattices as we glued more and more K_n's together. We found some explicit formulas as well as a recursion to calculate how many maximal chains are in certain types of bond lattices produced by gluing.

Primary Results

Let $\mathcal{MC}_{a,b,c}$ denote the number of maximal chains in the bond lattice of $G_{a,b,c}$. We proved the following.

1. $\mathcal{MC}_{a,1,c} = \dfrac{((a-1)c)!(a!)^c}{2^{(a-1)c}}$
2. $\mathcal{MC}_{3,2,c} = c!(2^{c+1}-1)$
3. $\mathcal{MC}_{4,3,c} = \dfrac{3}{2}(3^{c+2} - 2^{c+3} + 1)c!$
4. $\mathcal{MC}_{a,a-1,c} = (a-1)c\mathcal{MC}_{a,a-1,c-1} + \binom{a-1}{2}\mathcal{MC}_{a-1,a-2,c}$

Acknowledgments

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