Some Applications of Quotient Posets

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AMS Fall Sectional Meeting 2015
Special Session on Enumerative Combinatorics and Graph Theoretic Applications

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Three Applications

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1. Factorization of the characteristic polynomial

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2. Counting increasing forests

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2. Counting increasing forests
3. "Switching" the characteristic polynomial and the corank polynomial

Main Example

Let $G = (V, E)$ be a graph. A *flat* of G is a collection of edges F such that each connected component of the graph with vertex set V and edge set F is an induced subgraph of G.

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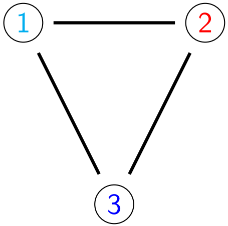

K_3

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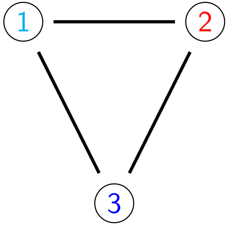

K_3

The flats are $\emptyset, \{12\}, \{13\}, \{23\}, \{12, 13, 23\}$.

Bond Lattice

The *bond lattice* of a graph G is the poset obtained by ordering the flats of a graph by containment. It is denoted $L(G)$.

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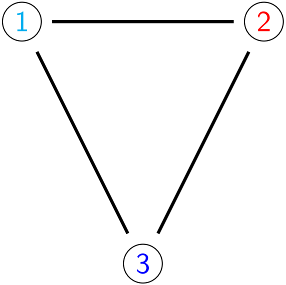
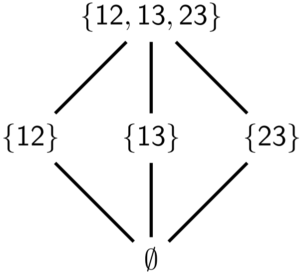


K_3



$L(K_3)$

The Möbius Function

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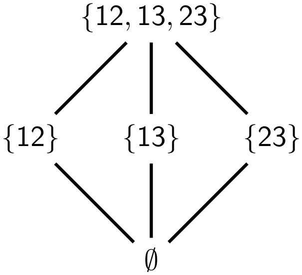
The *Möbius function* of a poset P is the unique map, $\mu : P \to \mathbb{Z}$, such that

$$\sum_{x \leq y} \mu(x) = \delta_{\hat{0},y}.$$

Example



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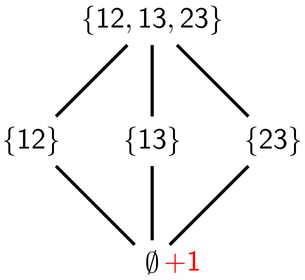

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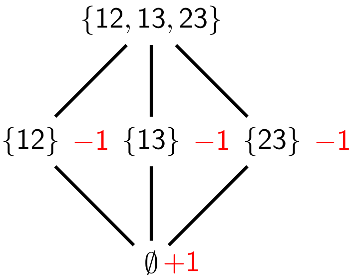

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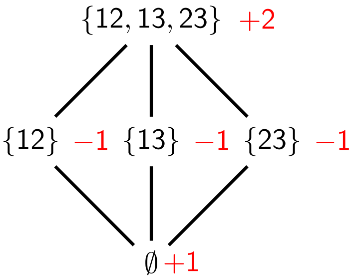

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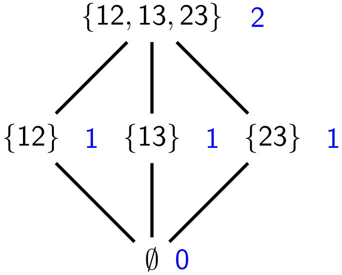

The Characteristic Polynomial

The *characteristic polynomial* of a poset P is

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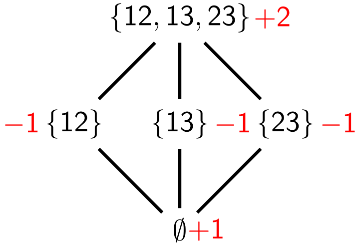

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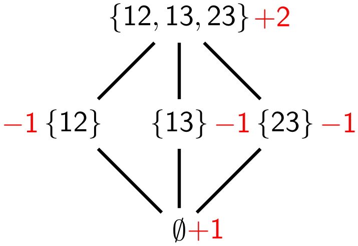

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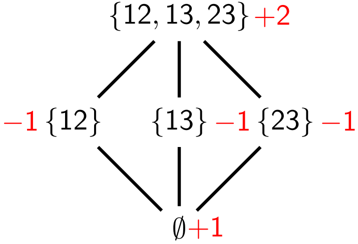

$$\chi(L(K_3),t) = t^2 - 3t + 2 = (t-1)(t-2)$$

If G is a graph, $c(G)$ is the number of connected components, and $P(G,t)$ is the chromatic polynomial of G then

$$P(G,t) = t^{c(G)} \chi(L(G),t).$$

First Application: Factorization

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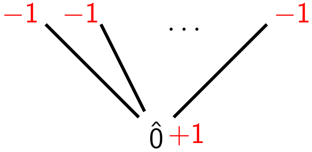
Lemma

Let P and Q be posets.

1. *If $P \cong Q$, then $\chi(P, t) = \chi(Q, t)$*
2. $\chi(P \times Q, t) = \chi(P, t)\chi(Q, t)$.

First Application: Factorization

The *n-claw* is the poset with a $\hat{0}$ and n atoms and is denoted CL_n.



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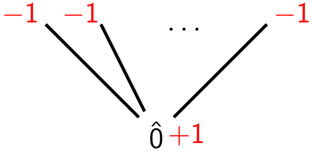

$$\chi(CL_n, t) = t - n.$$

First Application: Factorization

Since $\chi(L(K_3), t) = (t-1)(t-2)$, we may guess that $L(K_3) \cong CL_1 \times CL_2$.

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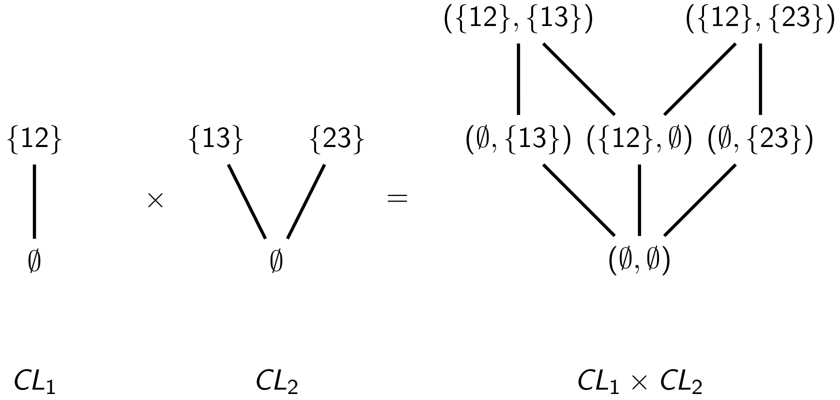

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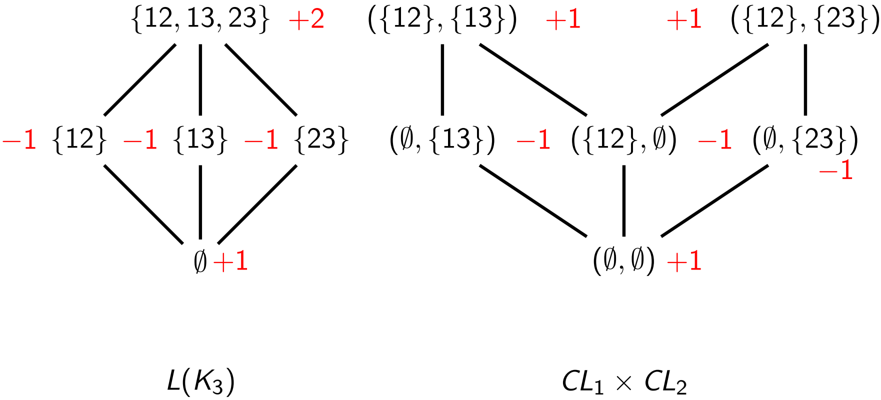

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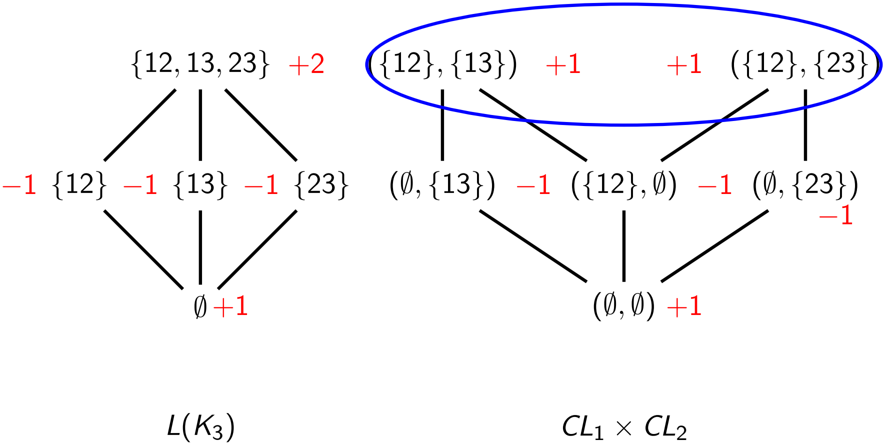

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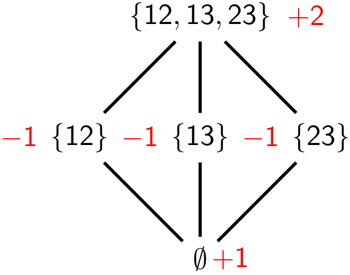
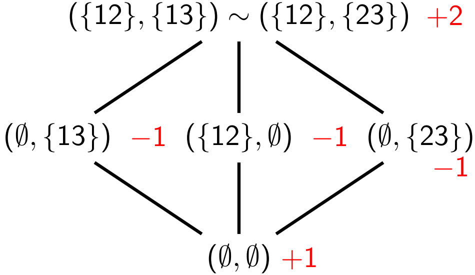
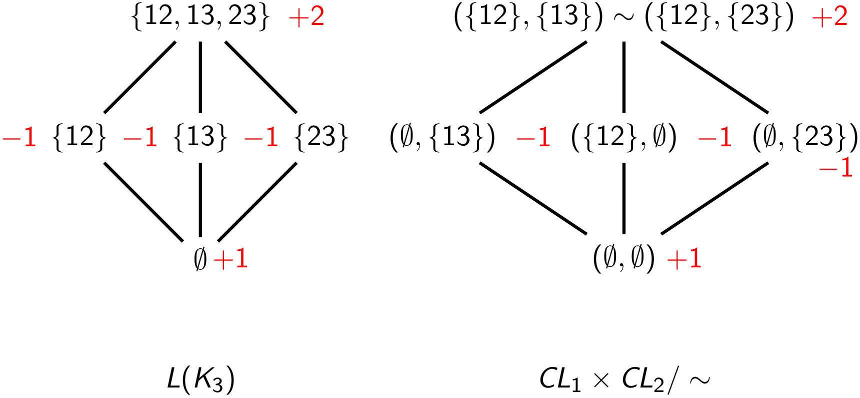


$L(K_3)$



$CL_1 \times CL_2 / \sim$

First Application: Factorization



Therefore,

$$(t-1)(t-2) = \chi(CL_1 \times CL_2, t) = \chi(CL_1 \times CL_2/\sim, t) = \chi(L(K_3), t).$$

Second Application: Counting Increasing Forests

Let T be a tree with vertices labeled by distinct integers. We say T is an *increasing tree* if the labels along any path starting at the minimum vertex of the tree form an increasing sequence.

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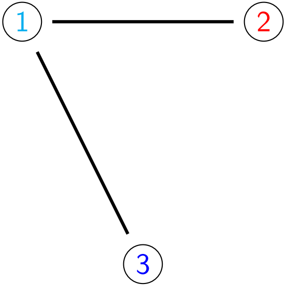
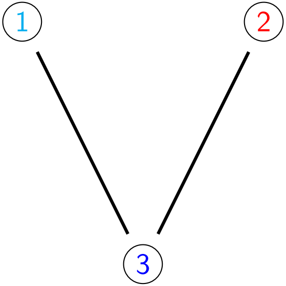
Example



Increasing Tree



Not An Increasing Tree

Second Application: Counting Increasing Forests

Let G be a graph with vertices labeled $1, 2, \ldots, n$. We say a collection F of edges of G is an *increasing forest of G* if each connected component of the graph with vertex set $1, 2, \ldots, n$ and edge set F is an increasing tree.

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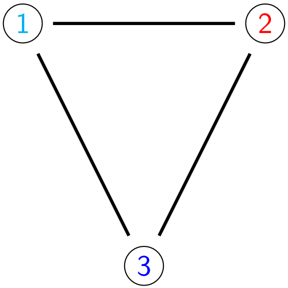

K_3

Increasing Forests of size 0: $\emptyset$

Increasing Forests of size 1: {12}, {13}, {23}

Increasing Forests of size 2: {12,13},{12, 23}

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Recall the products of claws that we considered when looking at $L(K_3)$.

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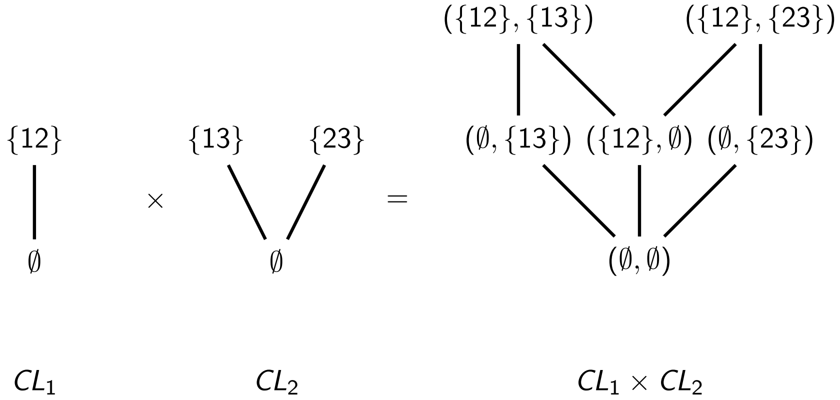

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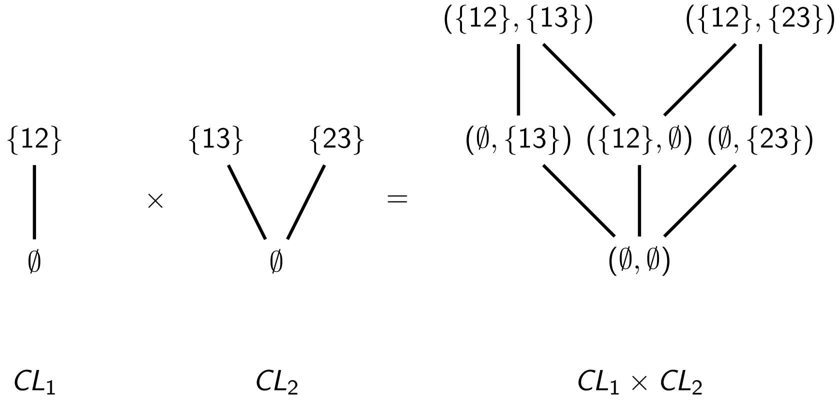

If one takes the union of the sets in each ordered pair, we get the 6 increasing forests of K_3.

Second Application: Counting Increasing Forests

The *increasing forest generating function* of G is given by

$$IF(G,t) = \sum_{k\geq 0} (-1)^k IF_k t^{n-k}$$

where IF_k is the number of increasing forests with k edges and $n = |V(G)|$.

Second Application: Counting Increasing Forests

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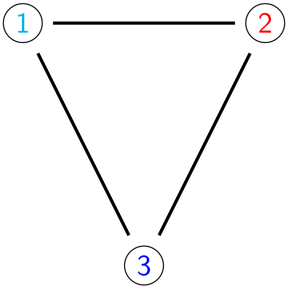

K_3

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So for K_3, we have

$IF(K_3, t) = t^3 - 3t^2 + 2t = t(t-1)(t-2) = P(G, t) = t\chi(L(K_3), t).$

Second Application: Counting Increasing Forests

A labeling of a graph with the numbers $1, 2 \ldots, n$ is called a *perfect elimination order* if whenever $i < j < k$ and ik and jk are edges then so is ij.

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Using quotient posets we can show the following.

Theorem

Let G be graph with vertices labeled $1, 2, \ldots, n$. We have

$$IF(G, t) = P(G, t)$$

if and only if the labeling is a perfect elimination ordering.

Third Application: The Corank Polynomial

Given a ranked poset P, the *corank polynomial* is defined by

$$CR(P, t) = \sum_{x \in P} t^{\rho(P) - \rho(x)}.$$

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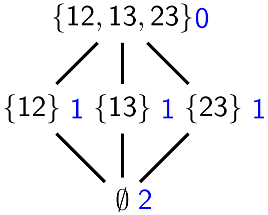
Example



$L(K_3)$

$$CR(L(K_3),t) = t^2 + 3t + 1$$

Third Application: The Corank Polynomial

For $L(K_3)$ we have

$$\chi(L(K_3), t) = t^2 - 3t + 2$$

and

$$CR(L(K_3), t) = t^2 + 3t + 1.$$

Is it possible to find another poset where these polynomials "switch"?

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Is it possible to find another poset where these polynomials "switch"?
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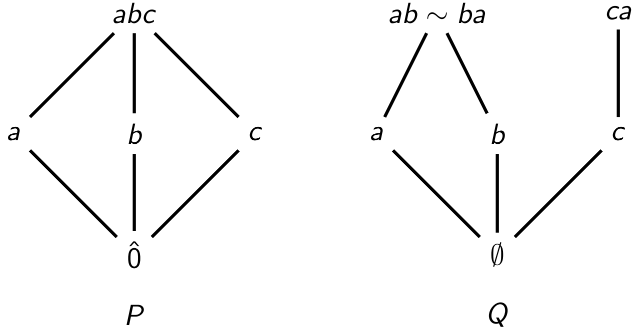
Third Application: The Corank Polynomial



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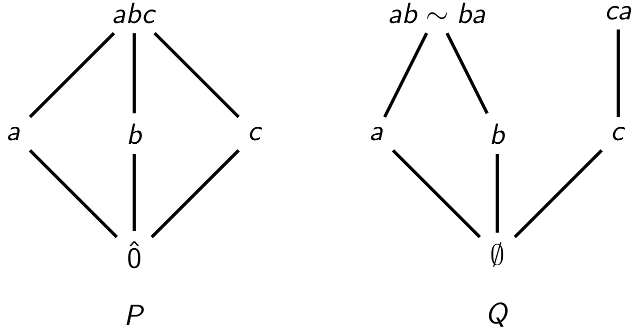

$$\chi(P,t) = t^2 - 3t + 2$$
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$$\chi(Q,t) = t^2 - 3t + 1$$
$$CR(Q,t) = t^2 + 3t + 2$$

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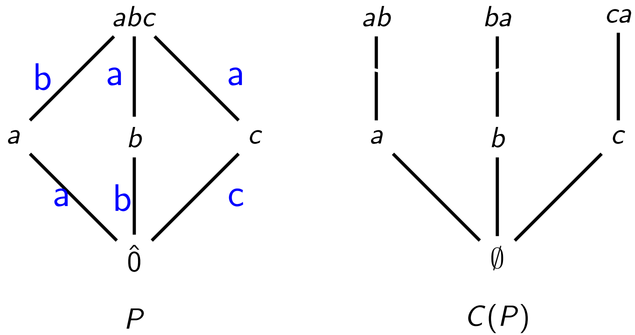
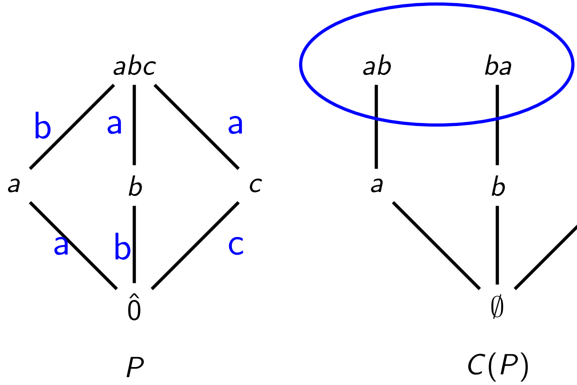


The edge labeling on P is an EL-labeling. The poset $C(P)$ is the poset of saturated chains starting at $\hat{0}$ in P.

Third Application: The Corank Polynomial

Identify elements of $C(P)$ which are rearrangements of each other.



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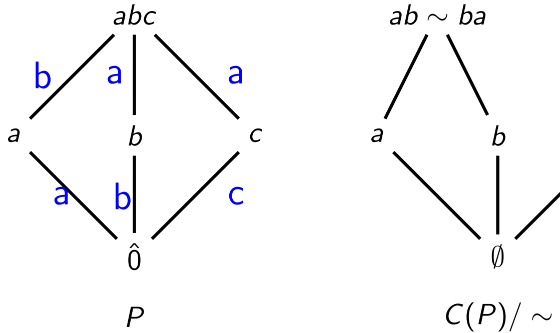
Third Application: The Corank Polynomial

Upon identification we get the following.



Thank You!