Parking Function Labelings of Noncrossing Bond Posets

Anna Salam and Joshua Hallam

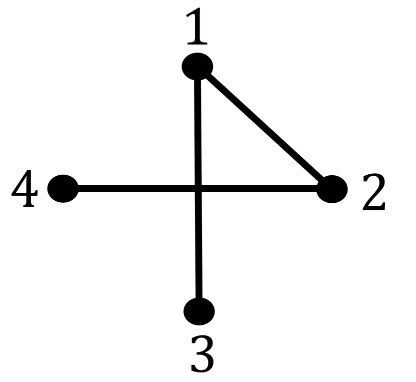
Mathematics Department | Loyola Marymount University | Los Angeles, CA

Research Question

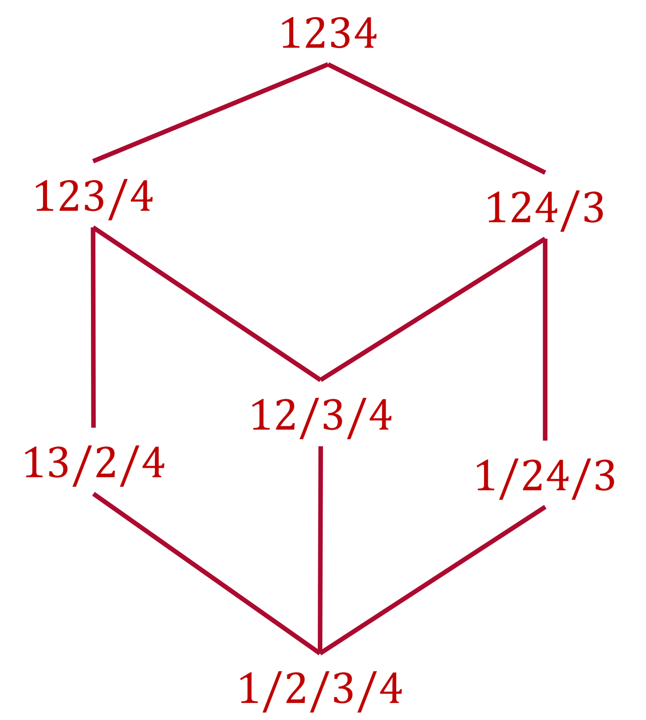
Which graphs will give rise to a noncrossing bond poset whose parking function labeling is a strict ER-labeling?

Posets & Labeling

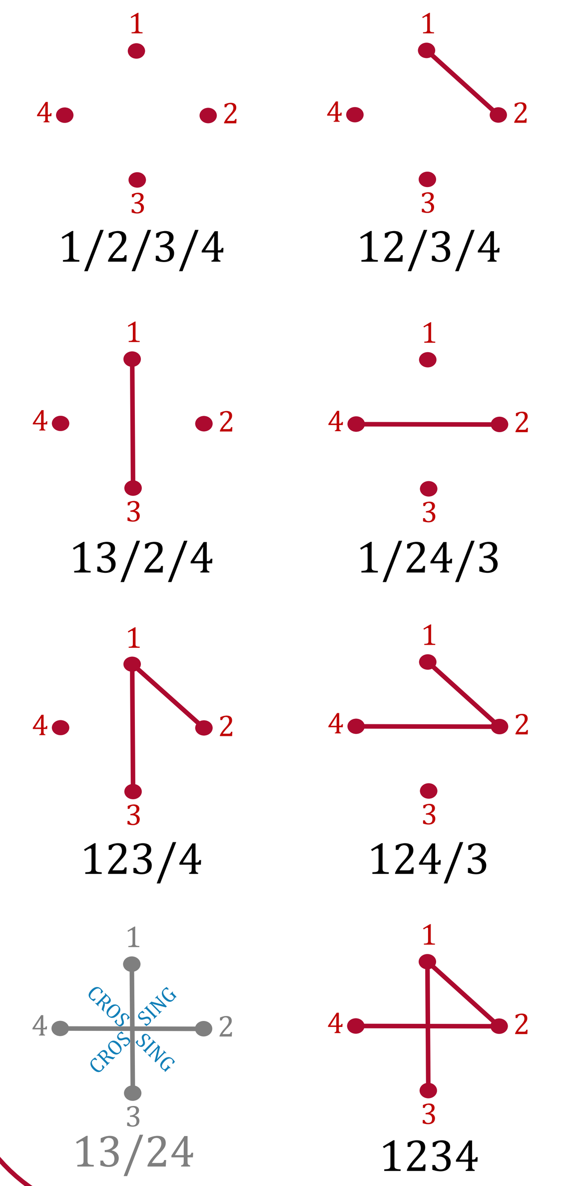
Example graph G :



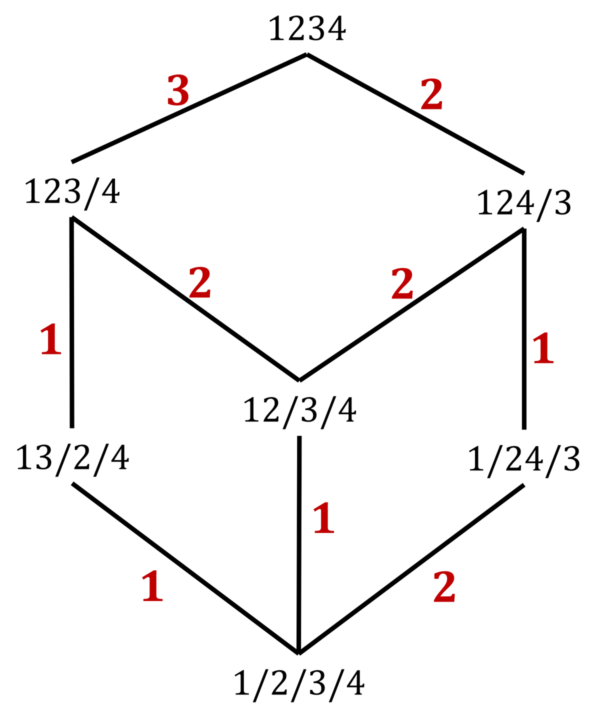
Noncrossing Bond Poset of G



Noncrossing Bonds of G



Parking Function Labeling of the Noncrossing Bond Poset of G



The parking function labeling of a poset is a strict ER-labeling if there exists a strictly increasing chain of parking function labels on every interval of the poset.

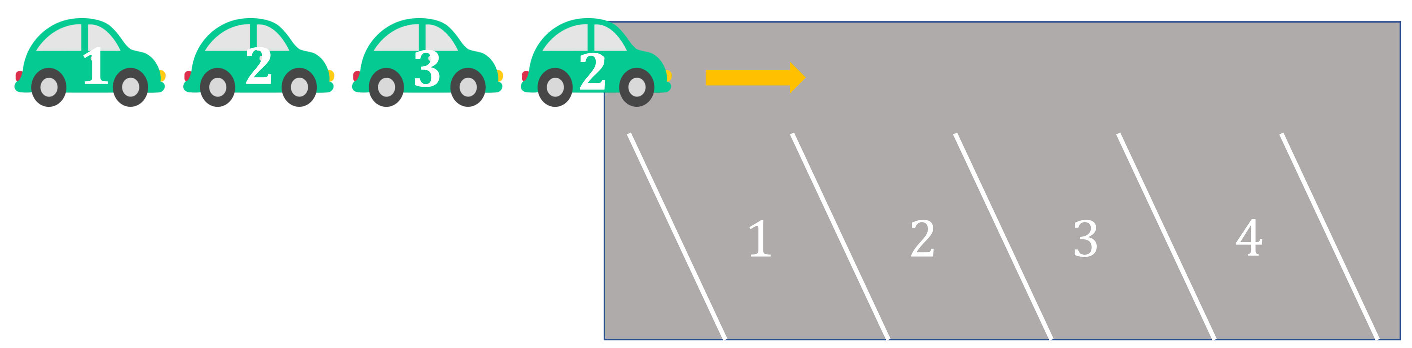
The parking function labeling of the noncrossing bond poset of G is a strict ER-labeling in the example above.

Parking Functions

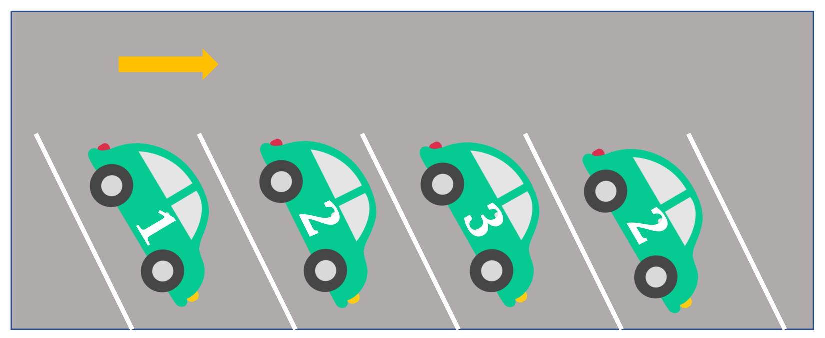
Consider a one-way street with parking spots $1, 2, 3, \ldots, n$ and the same number of cars each with a preferred parking spot. Cars go to their spots, one at a time, down the one-way street and park; if there is already a car in their spot, they park in the next available parking space.

The sequence of numbers corresponding to the cars' preferred parking spots is a parking function if every car can park.

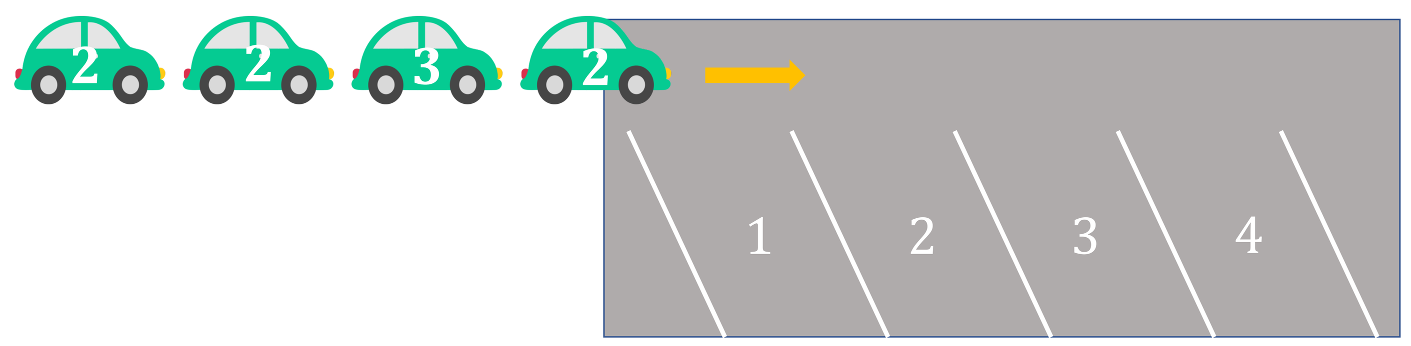
Example A: (2,3,2,1)



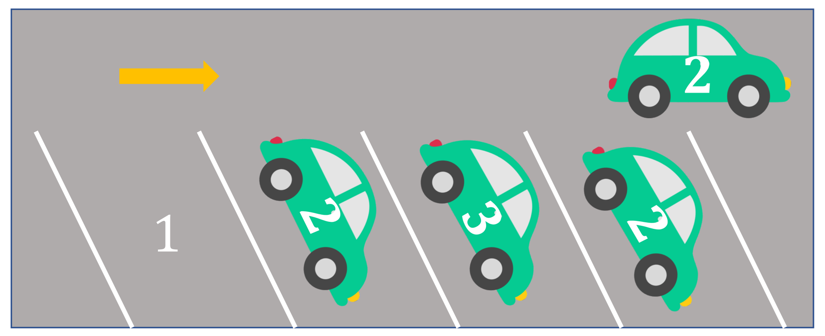
All cars can park, so the sequence (2,3,2,1) is a parking function.



Example B: (2,3,2,2)



Not every car can park, so the sequence (2,3,2,2) is NOT a parking function.

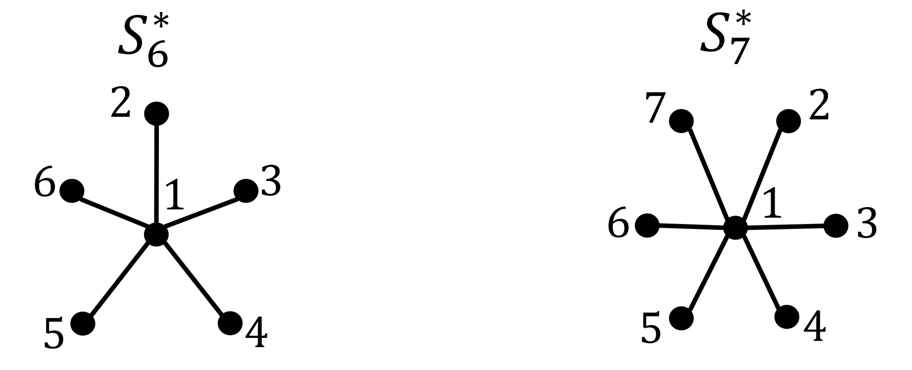


Every path from the bottom to the top of a noncrossing bond poset produces a sequence of parking function labelings that is a parking function itself.

This is significant because not every sequence of numbers is a parking function (as is the case in Example B).

Findings

Theorem: Let S_n^* be a star graph with 1 as its internal vertex, $n \epsilon \mathbb{N}$. $\forall\, n \epsilon \mathbb{N}, NC_{S_n^*}$ has a noncrossing bond poset whose parking function labeling is a strict ER-labeling.



Lemma: Merging the two blocks which produce the smallest PF label in an interval results in a noncrossing bond.

Lemma: After merging the two blocks that give the smallest PF label, the next label will always be greater.

Theorem: If the two blocks that give the smallest label of any grouping can be merged, then there is a strictly increasing chain of parking function labels.

Main Result

Let G be a graph and NC_G be the noncrossing bond poset of G. G is labeled with perfect elimination ordering if and only if NC_G has a parking function labeling that is a strict ER-labeling.

Perfect Elimination Ordering

The labeling of the vertices of a graph is a perfect elimination ordering (PEO) if whenever a vertex is connected to two smaller vertices, then those two vertices are also connected to each other.