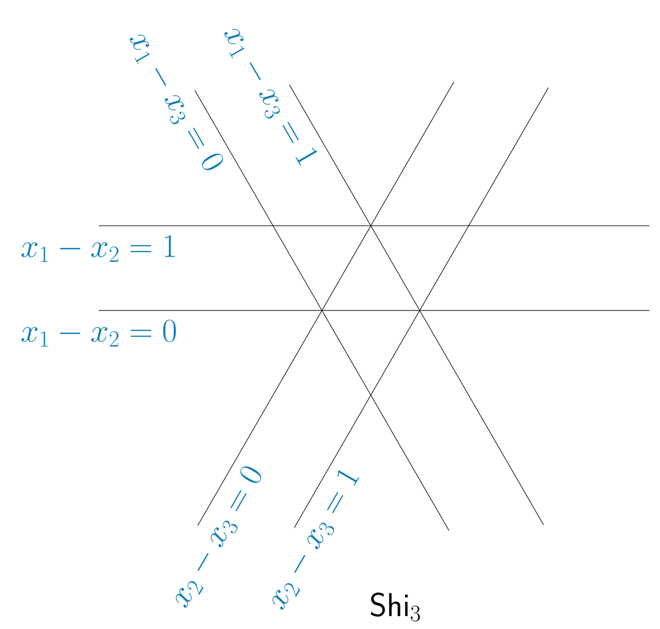
The Shi Arrangement and Pointed Partitions

Loyola Marymount University

Alex Abrams and Josh Hallam

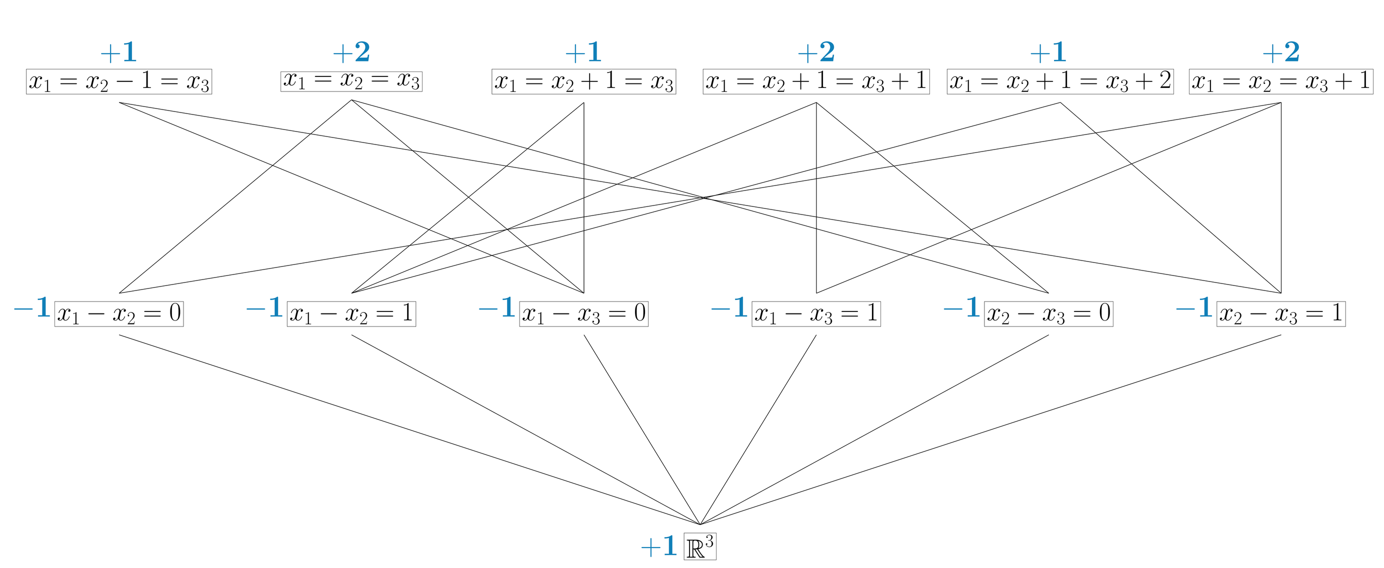
Hyperplane Arrangements

A **hyperplane arrangement** in $\mathbb{R}^n$ is a collection of (affine) vector spaces of dimension $n-1$. In $\mathbb{R}^2$, these are collections of lines and in $\mathbb{R}^3$ these are collections of planes. The **Shi arrangement**, Shi_n, is an important family of hyperplane arrangements. The figure below shows the Shi arrangement in $\mathbb{R}^3$ projected into $\mathbb{R}^2$.



The Intersection Poset

Given a hyperplane arrangement, we can look at all the possible intersections of the hyperplanes and then order the intersections by saying $W \leq V$ if and only if W contains V. This gives us the **intersection poset** of the hyperplane arrangement.



Intersection poset of Shi_3. Möbius values in blue.

The Möbius Function

The **Möbius function**, μ, is an important function defined on posets. Let $\hat{0}$ denote the minimum element of the poset. Then

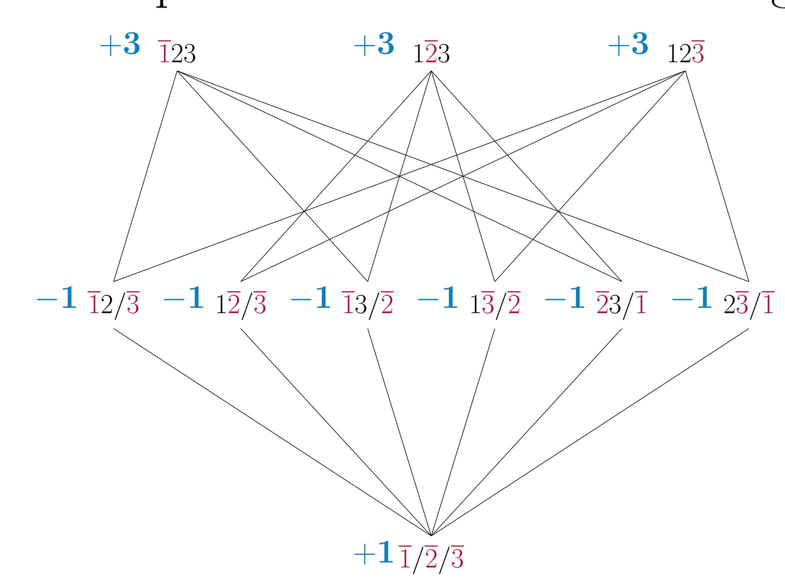
$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0}, \\ -\sum_{y<x} \mu(y) & \text{otherwise.} \end{cases}$$

The Möbius function has numerous applications.

Try it: Sum all the Möbius values of the intersection poset of Shi_3 and then count the number of small triangles in the picture of Shi_3. What do you notice? Now add the absolute value of the Möbius values. Can you find this number in the picture?

The Pointed Partition Poset

A **partition** of the set $\{1, 2, \ldots, n\}$ is a collection of subsets of $\{1, 2, \ldots, n\}$ which are pairwise disjoint and whose union is $\{1, 2, \ldots, n\}$. These subsets are called **blocks**. For example, $135/24/68/7$ is a partition of $\{1, 2, \ldots, 8\}$ with blocks $\{1, 3, 5\}$, $\{2, 4\}$, $\{6, 8\}$ and $\{7\}$. A **pointed partition** is a partition where each block has a special element which is **pointed.** We denote the pointed element by placing a bar over it. For example, $1\bar{3}5/\bar{2}4/6\bar{8}/\bar{7}$ is a pointed partition. The **pointed partition poset**, $\Pi_n^\bullet$, is the set of pointed partitions ordered so that $\pi \leq \sigma$ if σ can be obtained by merging blocks of π while retaining one of the pointed elements of the merged blocks.



Pointed partition poset $\Pi_3^\bullet$. Möbius values in blue.

The Characteristic Polynomial

The **characteristic polynomial** of a poset P is the polynomial given by

$$\chi(P, t) = \sum_{x \in P} \mu(x) t^{n-\rho(x)}$$

where n is the number of levels of P and $\rho(x)$ is the level that x is in. For example,

$$\chi(\mathsf{Shi}_3, t) = t^2 - 6t + 9 = (t-3)^2$$

and

$$\chi(\Pi_3^\bullet, t) = t^2 - 6t + 9 = (t-3)^2$$

The fact that the two posets have the same characteristic polynomial suggests a strong connection.

Primary Result

We proved the following.

Theorem: *The pointed partition poset* $\Pi_n^\bullet$ *is a quotient of the intersection poset of* Shi_n*. That is,* $\Pi_n^\bullet \cong \mathsf{Shi}_n/\sim$.

This was done by representing elements of Shi_n as directed graphs and collapsing certain elements together as explained in the next column.

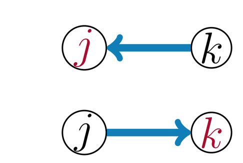
Quotient Posets

Let P be a poset and let $\sim$ be an equivalence relation on the elements of P. The **quotient poset**, $P/\sim$, is the poset obtained by ordering the equivalence classes of P under $\sim$. One can think of this as collapsing elements together in the diagram.

To show that the pointed partition poset is a quotient of the intersection poset of Shi_n, we represent elements of Shi_n as directed graphs as follows

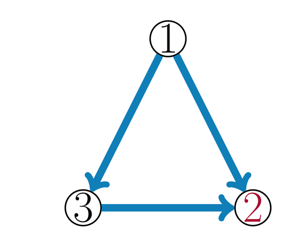
$x_j = x_k \longrightarrow$

$x_j = x_k + 1 \longrightarrow$

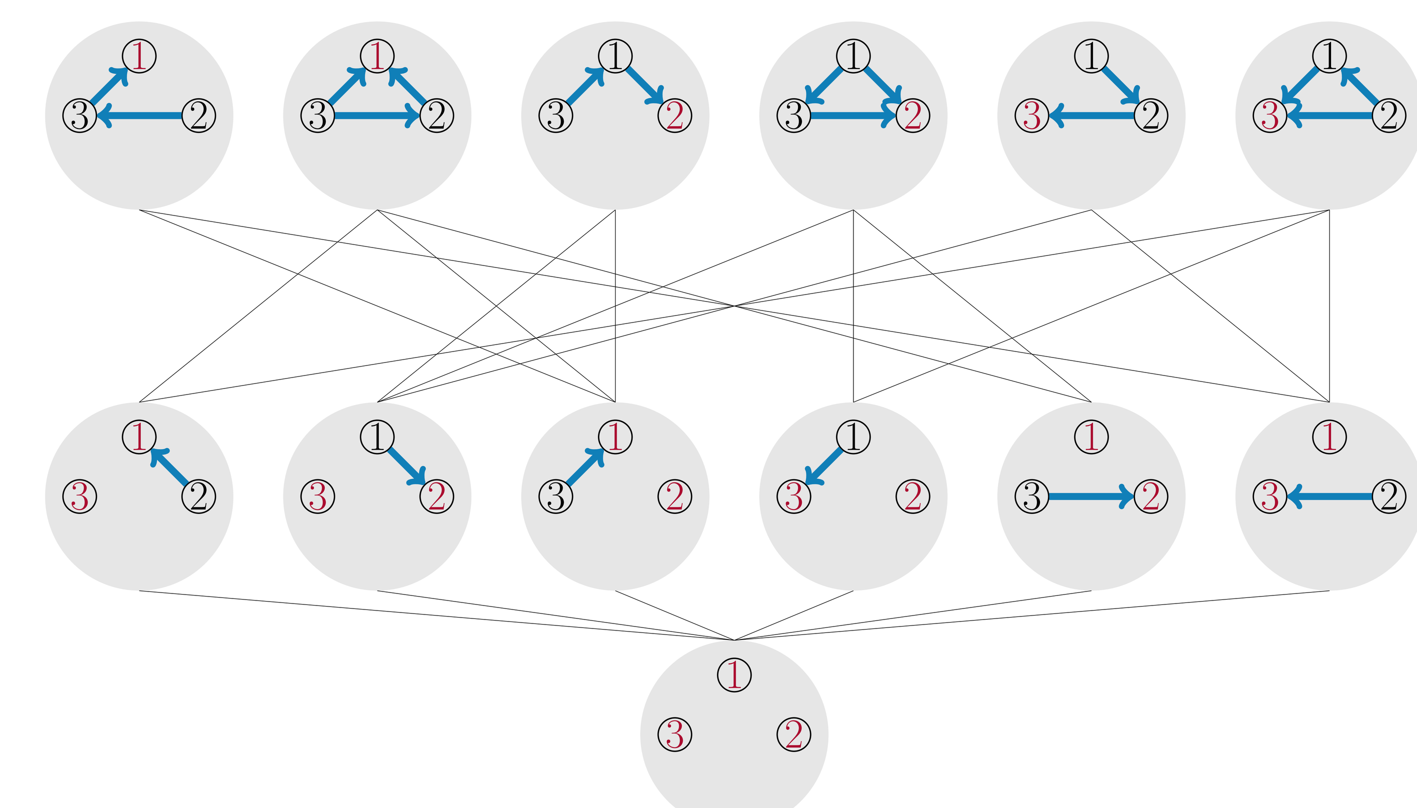


where $j < k$. For example, the following intersection of hyperplanes is represented as a graph below.

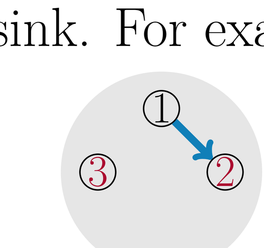
$$x_1 = x_2 + 1 = x_3 + 1$$
$$x_1 = x_2 + 1$$
$$x_1 = x_3 + 1$$
$$x_2 = x_3$$
$\longrightarrow$



Relabeling the elements of the intersection poset of Shi_3, we get the following representation.



We showed that these directed graphs are acyclic and each connected component has a unique sink. This gives us a map from Shi_n to $\Pi_n^\bullet$ where the graph gets mapped to the partition whose blocks correspond to connected components and the pointed element is the sink. For example,



$\longrightarrow$ $1\bar{2}/\bar{3}$

Collapsing graphs that correspond to the same pointed partition gives us our quotient.

Acknowledgments

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